Derivative Securities, Courant Institute, Fall 2010
http://www.math.nyu.edu/faculty/goodman/teaching/DerivSec10/index.html
Always check the class bboard on the blackboard site from home.nyu.edu (click on academics, then on Derivative Securities) before doing any work on the assignment.

Assignment 1, due September 15

Corrections: (Revised September 11 to clarify question 4)

1. A company needs to borrow money at time t to be repaid at time T but wants to fix the price of the loan today. The notation for this is that company agrees today (time $t = 0$) receive $F(0,t,T)$ (F is for "forward rate agreement") at time t and pay one unit of currency at time T. If Z is the dollar amount borrowed, the loan can be thought of as shorting $Z/F(0,t,T)$ units of the loan at time t. This produces $F(Z/F) = Z$ dollars at time t. The loan is repaid by paying $Z/F(0,t,T)$ dollars at time T to close the short position. The price of a loan agreed to and taken at time t is determined by $B(t,T)$ (B is for "bond".) The company may receive (or pay) $B(t_1,t_2)$ at time t_1 and pay (or receive) one unit at time $t_2 > t_1$. The difference between the $B(t,T)$ rate and the $F(0,t,T)$ rate is that the company must commit to the F rate at time $t = 0$, while it need not commit to the B rate until time t. In both cases, money changes hands only at times $t > 0$ and $T > t$.

   Explain an arbitrage argument that fixes $F(0,t,T)$ in terms of $B(0,t)$ and $B(0,T)$. Assume that the company can at time 0 take any position (positive or negative) in either or both of the $B(0,t)$ or $B(0,T)$ loans. The only constraint is that the company loan portfolio at time 0 should be *self financing*, which means that the total value of the position at time 0 is zero.

2. Give an arbitrage argument to show that the price of a vanilla European call or put is a convex function of the strike price, if the exercise time is held fixed. Assume that there are no market market frictions or trading constraints. *Discussion:* A function $f(K)$ is a convex function of K (in the interval $[K_0, K_3]$) if

$$f(\alpha K_1 + (1-\alpha)K_2) \leq \alpha f(K_1) + (1-\alpha)f(K_2)\ , \tag{1}$$

   whenever $0 \leq \alpha \leq 1$, and $K_0 \leq K_1 \leq K_2 \leq K_3$. If $f(K)$ is twice differentiable as a function of K, this is the same as $f''(K) \geq 0$. We can check (1), say, for $\alpha = 1/2$, by comparing the payout diagram for the portfolio:

$$\frac{1}{2}(\text{ call at } K_1\ ) + \frac{1}{2}(\text{ call at } K_2\ )$$

to the payout of the single call struck at $\frac{1}{2}(K_1 + K_2)$ (draw the diagram). The argument for other values of α and for puts is similar.

Some options dealers post a curve of options prices at which they are willing to buy and sell as functions of strike and maturity. If the curve is not convex for any reason (e.g. non-convex interpolation) this represents an arbitrage opportunity for the client.

3. Suppose X is a normal random variable with mean μ_X and variance σ_X^2. We write this as $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$. We will make constant use of the following results, particularly the formula $E\left[e^X\right] = e^{\cdots} \neq e^{\mu_X}$.

   (a) Calculate the mean and variance of $Y = e^X$. You may use the fact that if $Z \sim (0,1)$, then the probability density of Z is $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$, and that $\int_{-\infty}^{\infty} f(z)dz = 1$. Hint: Check that $X = \sigma_X Z + \mu_X$ has the $\mathcal{N}(\mu_X, \sigma_X^2)$ distribution, which is more than checking the mean and variance of X. Then write an explicit integral that is $E\left[Y\right] = E\left[e^X\right] = E\left[e^{\sigma_X Z + \mu_X}\right]$. To actually work the integral, you have to complete the square in the exponent. The same kind of thing works for $E\left[Y^2\right] = E\left[e^{2X}\right]$.

   (b) Show that $\text{var}(Y) = \sigma_Y^2 \approx \text{var}(X) = \sigma_X^2$ when μ_X and σ_X are close to zero. Interpret this in terms of the nature of the function $x \to e^x$ when x is close to zero.

4. This part asks you to do some numerical computations using Microsoft Excel. The file `ExcelHelp.pdf` on the class web site will help you do what this assignment asks for. This assignment addresses a frequently made point[1] that fund managers can skew their return distribution so that most years they outperform some benchmark, but some years they "blow up". If the probability of blow up is, say, 10%, the manager can have a ten year run of high returns. Getting fired when the fund blows up is a small price to pay for all those bonuses received in the mean time. How to "incentivize" managers not to do this is a lively topic among regulators, businesses, and investors.

   The geometric Brownian motion model of stock prices that we will discuss in class in a few weeks implies that S_T is a lognormal random variable that may be expressed as

   $$S = S_0 \exp(\sigma Z \sqrt{T} + (\mu - \sigma^2/2)T) \,, \qquad (2)$$

   This has the feature that $E[S_T] = S_0 e^{\mu T}$. The parameters μ and σ are the *expected rate of return* and the *volatility* respectively. According to the Black Scholes theory, (2) is the actual distribution of S_T that you would find in the market.

[1] See, e.g. *Fooled by Randomness* by Nassim Taleb.

The Black Scholes theory also says that the market price of a vanilla European call option, C, is

$$C = e^{-rT} E_{RN}[V(S_T)] \tag{3}$$

We have assumed a constant known risk free rate, so that $B_T = e^{-rT}$. The subscript RN above means that the expectation is taken in the *risk neutral measure*. That, in turn, means that one replaces μ with r in (2). Of course, the payout function is appropriate for the call option: $V(S_T) = (S_T - K)_+$. To summarize, you can find (an approximation to) C by evaluating the expectation value in (3) by Monte Carlo, but you must remember to use r instead of μ in (2). We will discuss the reason for this in a few weeks.

(a) The random number generator in Excel has proven (in my tests) to be inadequate for the steps below. Do not use it. Instead, download the file `randn.xls` from the class web site. It has 5000 standard normal random variables created by a decent generator.

(b) From now on, we will write just S instead of S_T. Use the standard normals to generate sets of a thousand independent lognormal stock prices, with vol parameter σ, expected rate of return μ, and time horizon T. Use these to make a Monte Carlo estimate of $E[S]$ Check that the mathematical relation $E[S] = S_0 e^{rT}$ is satisfied to within the accuracy of the Monte Carlo. If you have n independent standard normal random variables $Z_1, \ldots, Z_n$, you can make n independent lognormals using (2), use Z_k to make S_k. The Monte Carlo estimate of E[S] is $\frac{1}{n}\sum_{k=1}^{n} S_k$. Use $n = 1000$ here, $\sigma = .2$, $\mu = .1$, $S_0 = 100$, and $T = 2$. Create a histogram of the S_k to see what this particular lognormal distribution looks like. Since `randn.xls` has five sets of 1000 standard normals, you can repeat these steps five times to see how different the results are. This gives you an idea how large Monte Carlo errors are. You will learn much more about Monte Carlo computation like this going forward in the math finance program.

(c) Consider a vanilla European call option on S with strike price K that expires also at time T. Let C be the price of this option. Assume C is given by (3), and estimate the risk neutral expectation using Monte Carlo as above. Remember to use r in this part and μ when estimating $E[S]$. Use[2] $r = .03$ and $K = 130$.

(d) Consider a portfolio strategy that is to buy one share of stock and to sell short M call options at the initial time. The strategy holds the stock, invests the money from selling the option at the risk free rate. At time T, the portfolio value is $\Pi = S - MV(S) + MCe^{rT}$. With the parameters above, it is unlikely that the option will finish

[2] You know whether this risk free rate is correct for today's yield curve.

in the money ($V(S) > 0$). The most likely outcome is that the final value of the portfolio with the option is equal to $S + MCe^{rT}$. This outperforms the simple stock portfolio by MCe^{rT}. Use Monte Carlo to estimate the probability that $V(S) > 0$.

(e) Show (not by Monte Carlo) that $E[\Pi] < E[S]$. That means that although Π outperforms S with high probability its expected return is actually less.

(f) Create one graph that has histograms of the values of S and Π.

(g) Explain in a few sentences how this illustrates Taleb's point. The answer to this part will be graded for writing quality as well as correctness. Use complete sentences and correct grammar.