Assignment 11, due December 8

Corrections: (None yet)

1. Let $B(t, r, T)$ be the price of a zero coupon bond purchased at time t and redeemed at time T for one unit of currency if $r_t = r$ in the Vasicek model. Find the functions $C(t, t)$ and $D(t, T)$ so that

$$B(t, r, T) \;=\; \exp\left(C(t, T)(r_0 - r) + D(t, T)\right) \; . \tag{1}$$

   Do this by writing the PDE that B satisfies in the t and r variable (that's in the week 10 notes, or week 9) and using the formula (1) as an ansatz for the solution. Make sure to write clearly the ordinary differential equations (ODEs) that C and D satisfy and say what initial or final conditions C and D must satisfy. Then write the solutions to these ODEs. This is not the form of the ansatz we used in class and the notes, but it is not so different either.

2. Consider an option to pay K at time $T_1 < T$ in order to receive one unit of currency at time T. Suppose r_t satisfies the Vasicek process starting at r_0 today. What is the present price/risk neutral value of that option today? Give a Black Scholes type formula in terms of the constants $C = C(T_1, T)$ and $D(T_1, T)$. This means giving formulas for d_1 and d_2 in Black Scholes, which depend on the mean and variance of the gaussian random variable r_{T_1}. You can call them μ_{T_1} and $\sigma^2_{T_1}$ in your formulas for d_1 and d_2, but then you should give formulas for them too. Altogether, you should be able to create spreadsheet functions that evaluate the price in terms of the parameters a, $\overline{r}$, σ, r_0, C, D, K, T_1, and T. Warning, this is tricky and involves the numeraire stuff. You do not have to create the spreadsheet function, but the formulas should be enough to do so.

3. Suppose a bond has a risk neutral default intensity $\lambda(t)$ and the short rate is r_t. Suppose that the bond pays a continuous coupon $c\,dt$ in each interval dt until time T, when it pays the principal of one unit. Write a formula for the present value of the coupon and principal payments. Assume that you discount from time t to time 0 using the money market $M(t)$.

4. Suppose a bond pays a coupon $c\,dt$ in each interval of time dt up to time T and makes a principal payment of one unit at time T.

   (a) Find a formula for the yield to maturity if it is a par bond.

(b) Find a formula for the yield to maturity if the bond has price P today.

(c) Find a formula for the duration.

(d) What is the duration (in years) of a 30 year par bond with a $c = 5\%$ coupon.

(e) Suppose the risk free rate is a constant r between now and time T What coupon results in the bond being sold at par? Hint: this is very easy.

5. Suppose that all the zero coupon risk free bond prices $B(t, T)$ today at time $t = 0$. Suppose that Δt is one day, so that t_k represents the number of trading days from today. Read the argument in Hull that shows that forwards and futures have the same price if the risk free interest rate is known and fixed at r. Explain how the argument needs to be modified in case r is known but not constant. In particular, say what futures position you need to take on day k in order to replicate the forward at time T.