Derivative Securities, Courant Institute, Fall 2010

http://www.math.nyu.edu/faculty/goodman/teaching/DerivSec10/index.html

Always check the class bboard on the blackboard site from home.nyu.edu (click on academics, then on Derivative Securities) before doing any work on the assignment.

Assignment 2, due September 22

Corrections: (5 (h): $F(K)$ (the forward price) should have been $f(K)$, the probability density.)

1. This exercise gives a slightly different way to determine the risk neutral probabilities in the binary and binomial tree models. The *risk neutral world* is a world in which future prices are truly random and follow the risk neutral probabilities. We do not live in the risk neutral world, but in that world, the present price of any traded asset is equal to the discounted expected value of its future price. Probabilities in the risk neutral world are called the Q *measure*. Probabilities in the actual world are called the P *measure*. Expectation using the risk neutral probabilities is written $E_Q[\cdot]$. If f is the known market price of a traded asset today and V_T is the unknown price of that same asset "tomorrow", then

$$f \;=\; BE_Q\left[V_T\right] \; . \tag{1}$$

   (a) The value of cash today is 1. The value tomorrow is e^{rT}, in any state of the world tomorrow. Show that this means that the discount factor in (1) is $B = e^{-rT}$.

   (b) The price of a share of stock today is S_0 and the price tomorrow is S_T. Apply (1) to the asset price S. Assume that the possible values of S_T are uS_0 and dS_0. Show that this leads to formulas for the risk neutral probabilities $p_u = \Pr_{RN}\left(S_T = uS_0\right)$, and $p_d = \Pr_{RN}\left(S_T = dS_0\right)$.

   (c) It is particularly easy to find p_u and p_d if the future outcomes are symmetric with respect to the forward price $F_0 = S_0 e^{rT}$, which means that $uS_0 - F_0 = F_0 - dS_0$. Why?

2. In a one period binary model, suppose that $B(0,T) = .95$, $u = 1.1$, $d = .9$, and that $S_0 = 100$. Go through the steps of finding the arbitrage price of a put option with strike price equal to the forward price of the stock. This involves finding numerical values for p_u and p_d. Also determine the amount of stock and cash in the replicating portfolio.

3. Suppose instead that there are three possible prices tomorrow, uS_0, mS_0, and dS_0, with $u > m > d > 0$. Suppose also that there is a liquid exchange traded "call" option that pays one unit of currency if $S_T = uS_0$ and nothing otherwise. Let A_0 be the present price of this option.

(a) Under what conditions (values of u, m, d, A_0, e^{rT}) is this market complete?

(b) Under what conditions is this market arbitrage free?

(c) Explain the equations to be solved to determine the risk neutral probabilities if the market is complete and arbitrage free. If the equations are very complicated, you need not give the solutions explicitly.

4. A general discrete one period model is described by the matrix V that has the payouts of the N instruments in the M states of the world tomorrow. Row i of V has the payouts of instrument i in various states of the world tomorrow, V_{ij} for future state j. The spaces $\mathbf{L} \subset \mathbf{P} \subseteq \mathbb{R}^M$ are as defined in the notes for week 2.

   (a) Suppose that the market is arbitrage free but not complete. This means (see the notes) that $\mathbf{P}$ is a proper subset of $\mathbb{R}^N$ and that $\mathbf{L}$ does not intersect the set of non-negative payouts Q except at the origin. Show in this case that it is possible to add another row to V so that the extended $(N+1) \times M$ still represents an arbitrage free market and that the extended $\mathbf{L}$ is larger than the original $\mathbf{L}$. Hint: let u be a non-zero row vector with M components that is perpendicular to $\mathbf{P}$. Show that no such vector intersects Q (by picture if not by proof). Let the new instrument have payout structure u (pay u_j in state j) and have cost $f = 0$. Show that the expanded market is arbitrage free.

   (b) Using the result of part (a), show that any arbitrage free incomplete market is a subset of a complete arbitrage free market.

   (c) Use this to show that any arbitrage free market has a set of risk neutral probabilities. In class, we showed this only for complete arbitrage free markets.

   (d) (not an action item) This exercise is not so important for what we will do in the rest of the class, but it is a check that you have understood that part of the notes. I do not want to waste your time discussing stuff in class or the notes that you are not supposed to understand.

5. Suppose S_T can take any positive value. Suppose S_T has a risk neutral probability density $f(s)$. Suppose that $f(s) \to 0$ rapidly as $s \to 0$ and $s \to \infty$. Let $P(K)$ be the market price of a put option expiring at time T with strike price K. Let B be the discount factor. Let $F(K)$ be the price of a forward contract to sell the asset at time T for price K.

   (a) Show that $P(K) \to 0$ as $K \to 0$. Interpret this in view of the probability that the option will finish (expire) in the money.

   (b) Give a financial argument (which contract gives more rights) rather than a mathematical argument (which integral is larger) that $P(K) > F(K)$ for any K.

(c) Determine what happens to the spread $P(K) - F(K)$ as $K \to \infty$. Interpret this in terms of the relationship between a put and a forward contract if it is very likely that the put will be exercised.

(d) Suppose that there are market prices $P(K)$ for all K (a hypothesis Peter Carr is fond of). Using the integral expression for $P(K)$ in terms of $f(s)$, derive the formula

$$f(K) = \frac{1}{B} P''(K) = \frac{1}{B} \frac{d^2 P(K)}{dK^2} . \tag{2}$$

If f is computed from market prices in this way (or some other way), it is called the *market implied* probability density.

(e) Show that (2) is consistent with the convexity of $P(K)$.

(f) Show that if $P(K)$ is convex and has the large and small K behavior of parts (a) and (b), then (2) gives an $f(s)$ that is a probability density.

(g) Look up *butterfly spreads* in Hull (or wikipedia or wherever). Consider a butterfly spread of puts, one long at $K - \delta K$, one long at $K + \delta K$, and two short at K. Suppose there are enough liquid puts in the market that this butterfly makes sense with a relatively small difference in strikes, δK. Let D be the price of this butterfly. Show that $D/(\delta K)^2$ is a good approximation to $P''(K)$ in (2).

(h) Write D in terms of an integral of f against the payout of the butterfly. and interpret the fact that D approximately determines $f(K)$ in terms of the relationship between the butterfly payout and the delta function.

6. No computing this week.