Derivative Securities, Courant Institute, Fall 2010

http://www.math.nyu.edu/faculty/goodman/teaching/DerivSec10/index.html

Always check the class bboard on the blackboard site from home.nyu.edu (click on academics, then on Derivative Securities) before doing any work on the assignment.

Assignment 3, due September 29

Corrections: (Sept 27: Problem (3) given the correct formula. I also corrected the formula in the notes.)

1. This question is to be done by hand, possibly with a calculator. It is very important that you understand the workings of the binomial tree model, so this will be the simplest computation that illustrates the main features of binomial tree pricing and dynamic replication. Assume there are 4 discrete times, three stages of the tree, and four possible final prices. Take the present spot to be $S_0 = 100$. Use the other parameters $B = e^{-r\delta t} = .9$, $u = 1.2$, and $d = .8$. The four possible prices at the final time are $S_{T,0} = S_0 d^3$, $S_{T,1} = S_0 d^2 u$, $S_{T,2} = S_0 d u^2$, and $S_{T,3} = S_0 u^3$.

   (a) Compute the risk neutral probabilities p_u and p_d.

   (b) Assume the option is a vanilla European put with strike price $K = 100$ expiring at the final time, fill in the binomial tree with option prices. This involves six computations.

   (c) What is the binomial tree price of the option at the initial time?

   (d) Does this price make sense, given the relationship between the strike price and the forward price?

   (e) Compute the values of Δ, the amount of stock to hold in the replicating portfolio, at each internal node of the tree. This is another six computations.

   (f) Suppose you start at time t_0 with an amount of cash equal to the computed option price, and that you want to construct and maintain the stock/cash dynamic replicating portfolio for the option. Suppose the stock price trajectory is $S_0 \to uS_0 \to duS_0 \to d^2uS_0$. Describe the transactions that you must do at the initial time and at the two intermediate times. Show that at each time along this path, the total value of your stock/cash replicating portfolio is equal to the option value from part b. This will be easy if you use the results from part e.

   (g) Repeat part f with the path $S_0 \to dS_0 \to d^2S_0 \to d^2uS_0$. Comment on the differences between the transactions in the two cases and the differences in the result.

2. Use (6) on page 3 of the Week 3 notes to calculate $E_Q[S_n^2]$. It may be helpful to use the binomial formula $(a+b)^n = \sum \binom{n}{j} a^j b^{n-j}$. This is the binomial version of the calculation of $E[S^2]$ in the lognormal model from homework 1.

3. Derive the formula (10) on page 4 of the Week 3 notes. The original formula in the notes was wrong. The correct formula is

$$E[A \mid \mathcal{F}_k] \;=\; \sum_{i=0}^{k-1} S_i \;+\; \frac{\frac{1}{B^{n-k}} - 1}{\frac{1}{B} - 1} S_k$$

4. Give a formula for the value of r_u that re-weights the Q measure so that S_k becomes a martingale. The answer may depend on B, u, and d.

5. Find the solution of (21) and (22) (page 10 of the Week 3 notes) accurate to order δt, when δt is small. For example, if $\alpha = b\sqrt{\delta t} + c\delta t + d\delta t^{3/2}$, then $\alpha \approx b\sqrt{\delta t} + c\delta t$ is accurate to order δt. Use $B = e^{-r\delta t}$. Assume $p_u = p_d = \frac{1}{2}$.

6. Download the file `bintree.cpp`. It is a C++ program that is a partial implementation of a binomial tree. If you compile and run it, it should produce a file called `option.csv`.

   (a) Download the file `option.csv` from the class web site and check that it is the same as the one the program produced for you. A file in `.csv` format ("csv" is for *comma separated values*) is meant to be read by Microsoft Excel. Check that this works for you.

   (b) Read over the program. You will see that it is incomplete in several crucial places. These are marked with comments starting

   ```
   // **** FIX
   ```

   Most importantly, the actual computation of p_u and p_d has been left out. Modify the program to calculate the correct p_u and p_d then run it with the correct parameters to compare with your hand calculations from question 1. You may be tempted to do these tasks in the opposite order. Please resist this temptation.

   (c) Modify the program to use the somewhat more realistic parameters $B = .99$, $u = 1.02$, $d = .98$, $n = 20$. Use Excel to make a plot of $P(K)$ for a range of K values that shows the important behavior of $P(K)$. You will have to choose the number of K values to use and the range of values. The procedure will be to run the C++ program then read the `option.csv` file into Excel for plotting[1]. Check that $P(K)$ has the qualitative behavior for large and small K predicted in assignment 2.

[1] In a more perfect world, one could call the C++ program directly from Excel and automate the whole process. Microsoft makes this difficult.