Derivative Securities, Courant Institute, Fall 2010

http://www.math.nyu.edu/faculty/goodman/teaching/DerivSec10/index.html

Always check the class bboard on the blackboard site from home.nyu.edu (click on academics, then on Derivative Securities) before doing any work on the assignment.

Assignment 5, due October 13

Corrections: (none yet)

1. This gives a simple derivation of the Black Scholes formula for the Delta of a put. We start with the following fact from calculus. If
$$I(s) \; = \; \int_{-\infty}^{Z_*(s)} G(z,s)\, dz \; .$$
then
$$\frac{d}{ds} I(s) \; = \; G(Z_*(s)) \frac{dZ_*}{ds} \; + \; \int_{-\infty}^{Z_*(s)} \partial_s G(z,s)\, dz \; .$$
Apply this formula to the integral (16) from the week 4 notes and use an identity about the N function to derive (2) from week 5.

2. Use the Black Scholes call formula to show that when $s = K$ the call price has the short time approximation
$$C \; = \; K\sigma\sqrt{T} N'(0) \; + \; O(T) \quad \text{as} \quad T \to 0 \; . \tag{1}$$
Note that the $C = 0$ when $s - K$ and $T = 0$. Show that this implies that the "velocity" of C as a function of T goes to infinity as $T \to 0$. This agrees with the qualitative behavior of Θ as $T \to 0$. Show that the agreement is quantitative, in that the derivative of the approximation $C \approx K\sigma\sqrt{T}N'(0)$ matches (i.e. agrees to leading order with) Θ, at the money for short time.

3. This exercise asks you to create the spreadsheet infrastructure to compute the Greeks of portfolios of options.

   (a) Start with the macros from assignment 4 and add new ones that compute the Greeks (Delta, Gamma, Vega, Theta) of vanilla European puts and calls. These should have names related to the macros from assignment 4.

   (b) Create a spreadsheet that tests these macros by comparing their results to finite difference approximations to the derivatives of the put and call values. For example, you could test Delta by computing $[C(s + \delta s) - C(s - \delta s)]/(2\delta s)$. Assemble these into a compact easy to understand spreadsheet with parameters $(S, K, \sigma, T, \delta s, \delta t, \delta\sigma, r)$

easy to identify and change. The finite difference values and the exact formula values should be next to each other or easy comparison. Hand in a printout or a screen capture of the spreadsheet.

(c) Now create a spreadsheet for computing the total Greeks for portfolios of options. Make one set of columns for entering puts with their parameters and portfolio weights and another for entering calls. Behind each of these, put the prices and corresponding Greeks. Sum the prices and Greeks in each column (with appropriate weights) and combine put and call Greeks to get the overall value and Greeks for the portfolio.

4. One might think that a good way to make money using options would be to find an inexpensive portfolio with a large positive Θ. After all Θ is the time derivative of the portfolio value, so a large positive Θ means rapid increase in value. Use the spreadsheet from question 3 to construct a sensible[1] large positive Theta but inexpensive portfolio. Use the other Greeks of this portfolio to comment on it as a safe way to make money.

5. This question uses the explores the convergence of random walk to Brownian motion and the distribution of a path dependent function, the maximum. Download the C++ program `PathMax.cpp` run it. This produces an output file called `Max.csv`. Your `Max.csv` file should be identical to the one posted. The max function is

$$V(X_{[0,T]}) \;=\; \max_{0\leq t\leq T} X_t \; .$$

For a Brownian motion, the *Kolmogorov reflection principle* states that[2]

$$\Pr(V(X_{[0,T]}) \geq z) \;=\; 2\Pr(X_T \geq z) \; .$$

The left side is given in terms of N functions.

Modify the program `PathMax.cpp` to simulate a random walk that moves up by a with probability 2/3, and down by $2a$ with probability 1/3. In the notation of the week 4 and week 5 notes, this is $X^{\delta t}_{t_{j+1}} = X^{\delta t}_{t_j} + Z^{\delta t}_j$, where $Z^{\delta t}_j = a$ or $Z^{\delta t}_j = -2a$. Choose a as a function of δt so that $X^{\delta t}_{[0,T]}$ converges in distribution to standard Brownian motion.

Take $T = 1$ and a several δt values tending toward zero and see that the probability distribution of $V(X^{\delta t}_{[0,T]}$ converges to the distribution of $V(X_{[0,T]}$ This amounts to some algebra, then comparing the computed histogram to an appropriate Gaussian. You can compute the Gaussian values at the bin centers, which are available in the spreadsheet.

---

[1] This means using options not too far out of the money or in the money that they would not trade in real markets, reasonable values of σ and T, etc.

[2] This is simple to understand. If X touches the value z, there is a first time it touches, τ. Starting at $X_\tau = z$, X_T has a 50/50 chance to be above z, by symmetry. Symmetric random walks also satisfy this symmetry principle, but not asymmetric ones.

6. Let $f(x,y)$ be a smooth function with a Taylor series

$$\begin{aligned} f(x+\delta x, y+\delta y) \;&=\; f(x,y) \,+\, \partial_x f(x,y)\delta x \,+\, \partial_y f(x,y)\delta y \\ &\quad + \frac{1}{2}\partial_x^2 f(x,y)\delta x^2 \,+\, \partial_x\partial_y f(x,y)\delta x\delta y \,+\, \frac{1}{2}\partial_y^2 f(x,y)\delta y^2 \\ &\quad O\left(\left|(\delta x,\delta y)\right|^3\right) \;. \end{aligned}$$

Suppose that X_t and Y_t are independent standard Brownian motions and that

$$R_t \;=\; \left(X_t^2 + Y_t^2\right)^{1/2} \;.$$

Show that R_t is a diffusion (as long as $R > 0$) and calculate the infinitesimal mean and variance in terms of R_t in the sense of (18) and (19) of week 5. Calculate $a(R)$ and $b(R)$. That is, compute $E[\delta R \mid \mathcal{F}_t]$ and $\mathrm{var}(\delta R \mid \mathcal{F}_t)$ and keep terms of order δt or larger only. Believe it or not, this *Bessel process* comes up in finance.