Derivative Securities, Courant Institute, Fall 2010
http://www.math.nyu.edu/faculty/goodman/teaching/DerivSec10/index.html
Always check the class bboard on the blackboard site from home.nyu.edu (click on academics, then on Derivative Securities) before doing any work on the assignment.

Assignment 6, due October 20

Corrections: (none yet)

1. Verify by explicit calculation that the right side of the formula $\int_0^t W_t dW_t = \frac{1}{2}W_t^2 - \frac{1}{2}t$ is a martingale. This is more than $E\left[\frac{1}{2}W_t^2 - \frac{1}{2}t\right] = 0$.

2. The Black Scholes equation (derivation at the end of week 6 notes) implies that a vanilla European call option satisfies

$$\Theta_C + \frac{s^2\sigma^2}{2}\Gamma_C + rs\Delta_C - rC = 0 \,.$$

   Verify this using the explicit formulas for the Greeks.

3. Let $I^n = \sum_{k=0}^{n-1} f_{t_k}(X_{t_{k+1}} - X_{t_k})$ as in the week 6 notes. Take $T = n\delta t$. Write the expression for $R^n = I^{2n} = I^n$ in the case $f_t = S_t$ and $X_t = S_t$, where $dS_t = \mu S_t dt + \sigma S_t dW_t$. This is just the formula in the notes for this case. Find an approximation

$$E\left[(R^n)^2\right] \approx \delta t\, E\left[\int_0^T A_t dt\right] \,.$$

   Here A_t is a simple function of S_t that you will find as part of the solution.

4. Suppose $S_0 = X_0 = 1$, and suppose that X_t is found from S_t using

$$\frac{dX_t}{X_t} = 2\frac{dS_t}{S_t} \,. \qquad (1)$$

   This is a model of a "leveraged" fund, that returns twice some other asset returns.

   (a) Show that if S is a differentiable function of t than $X_t = S_t^2$. Hint: use ordinary calculus and calculate $\frac{d(S_t^2)}{S_t^2}$, or calculate $d\log(X_t)$ and $d\log(S_t)$ and use (1).

   (b) Show that if S_t is a diffusion process then (1) does not imply that $\log(X_T) = 2\log(S_T)$. Find the correct formula for $\log(X_T)$ in terms of $\log(S_T)$ and a Riemann integral.

   (c) Evaluate this formula explicitly in the case S_t is a geometric Brownian motion.

5. Suppose $S_{1,t}, \ldots, S_{n,t}$ are independent geometric Brownian motions with expected return rates μ_i and volatilities σ_i (so $dS_{i,t} = \mu S_{i,t}dt + \sigma S_{i,t}dW_{i,t}$, where the W_i are independent Brownian motions). Let

$$X_t = \sum_{i=1}^{n} w_i S_{i,t}$$

be a portfolio of these assets with portfolio weights w_i. Is it true that X_t is a geometric Brownian motion? Why or why not?

6. No computing this week (not for this class anyway).