Assignment 7, due October 27

Corrections: (Formulas (1) and (4) corrected to take into account the drift term, Formula (5) corrected to replace P by Q on the left and remove the weight term)

1. *Girsanov's formula* is a reweighting formula for diffusions similar to the reweighting we did for the binomial tree process we did earlier. We will use this when we talk about changing "worlds" (i.e. measures) when talking about interest rate modeling. This exercise walks you through one approach to this.

   (a) Suppose X_t is a diffusion process, with respect to a probability measure P on path space, with $E_P[dX_t \mid \mathcal{F}_t] = a(X_t)dt$ and $E_P[(dX_t)^2 \mid \mathcal{F}_t] = b^2(X_t)dt$. Consider the processes $Y_t = \int_0^t M(X_s)dX_s$ and $Z_t = e^{Y_t} = e^{\int_0^t M(X_s)dX_s}$. Compute dY_t and $(dY_t)^2$ in terms of dX_t and $(dX_t)^2$, then compute dZ_t and $(dZ_t)^2$ in terms of dY_t and $(dY_t)^2$. Put these together to get expressions for $E_P[dZ_t \mid \mathcal{F}_t]$ and $E_P\left[(dZ_t)^2 \mid \mathcal{F}_t\right]$ in terms of $M(X_t)$, $a(X_t)$, and $b(X_t)$. Normally we do not bother to write $E_P[\cdot]$ when we are talking about only one measure.

   (b) Replace the above with the more complex expression

   $$Y_t = \int_0^t M(X_s)\,dX_s - \tfrac{1}{2}\int_0^t M^2(X_s)b^2(X_s)\,ds - \int_0^t a(X_s)ds \ , \quad (1)$$

   and calculate $E_P\big[\,d\,e^{Y_t} \mid \mathcal{F}_t\big]$. Use this to show that $E_P\big[\,e^{Y_t}\big] = 1$ for all t, and that $E_P\big[\,e^{Y_T} \mid \mathcal{F}_t\big] = e^{Y_t}$.

   (c) Changing notation, define the weight function

   $$L(X_{[0,T]}) \ = \ e^{Y_T} \ . \quad (2)$$

   In the above, $X_{[0,T]}$ is the whole path between times 0 and T, and I write $L(X_{[0,T]})$ to emphasize that the weight factor L is a function of the whole path. Define a reweighted "measure", Q, by

   $$E_Q\left[\,V\big(X_{[0,T]}\big)\right] \ = \ E_P\left[\,V\big(X_{[0,T]}\big)\,L\left(X_{[0,T]}\right)\right] \ . \quad (3)$$

   We have not defined measures, but we can check that the definition (3) has the properties measure like properties that $E_Q[1] = 1$, and that if $V\big(X_{[0,T]}\big) \geq 0$ for all paths $X_{[0,T]}$, then $E_Q\left[\,V\big(X_{[0,T]}\big)\right] \geq 0$. Do these checks.

(d) Define the weight function starting at time t as

$$L\big(\, X_{[t,T]}\big) \;=\; \exp\!\left(\int_t^T M(X_s) dX_s \;-\; \tfrac{1}{2}\int_t^T M^2(X_s) b^2(X_s) ds \;-\; \int_t^T a(X_s) ds \right) . \tag{4}$$

Clearly $L\big(\, X_{[0,T]}\big) = L\big(\, X_{[0,t]}\big)\, L\big(\, X_{[t,T]}\big)$. The following formula will become obvious:

$$E_Q\big[\, V\big(\, X_{[t,T]}\big) \mid \mathcal{F}_t \big] \;=\; E_P\big[\, V\big(\, X_{[t,T]}\big)\, L\big(\, X_{[t,T]}\big) \mid \mathcal{F}_t \big] \;, \tag{5}$$

where $V\big(\, X_{[t,T]}\big)$ refers to a function of the path that depends only on values between t and T. I did not find a non-technical way to show this, so I ask you just to accept it, and to verify it in the case $V\big(\, X_{[t,T]}\big) = 1$. This is not hard, but it is not nothing either. You need to use some things from earlier parts.

(e) Find simple formulas for $E_Q[\, dX_t \mid \mathcal{F}_t]$ and $E_Q\Big[\, (dX_t)^2 \mid \mathcal{F}_t \Big]$ in terms of $a(X_t)$, $b(X_t)$, and $M(X_t)$. Also note that all paths are continuous in the Q measure because reweighting does not change the fact that a path is continuous.[1] Use these to support the statement that if $X_{[0,T]}$ is a diffusion in the P measure, then it also is a diffusion in the Q measure.

(f) What is the relation between the noise coefficient of the diffusion in the P and Q measures? It is a harder but true part of the Girsanov story that you cannot change the diffusion coefficient of a diffusion by reweighting of any kind.

(g) Find the reweighting function $M(x)$ that changes the drift from $a(x)$ to $\overline{a}(x)$. Note that this works only if $b \neq 0$.

(h) Write explicitly the Girsanov reweighting formula that changes $dS_t = \mu S_t dt + \sigma S_t dW_t$ to $dS_t = r S_t dt + \sigma S_t dW_t$. Let the notations $E_r[\cdot]$ and $E_\mu[\cdot]$ refer to expectations in these two measures. Find the function $L\big(\, S_{0,T]}\big)$ so that $E_r[V\big(\, S_{0,T]}\big)] \;=\; E_\mu[V\big(\, S_{0,T]}\big)\, L\big(\, S_{0,T]}\big)]$ for every function of the path $V\big(\, S_{0,T]}\big)$.

2. Suppose you have to pay a tax gdt for each share of stock you own during the time interval $(t, t+dt)$. If you own M shares of stock, your tax is $Mgdt$. This includes being subsidized for short positions.[2] Go through the derivation of the Black Scholes equation to see how this tax changes the equation. Do the same for the more realistic tax that asks you to pay $M\, |g|\, dt$ instead.

3. Suppose $S_{1,t}$ and $S_{2,t}$ are two independent stocks with expected growth rates μ_1 and μ_2 and volatilities σ_1 and σ_2 respectively. Consider a *remorse* option that allows the holder to trade one share of S_1 for one share of S_2 at time T if she or he wishes.

[1] There is some measure theory statement here being left out.

[2] Probably a real tax law would not do this.

(a) Write a formula using the $(\cdot)_+$ function for the final time payout for remorse.

(b) Give a hedging argument that constructs a PDE for the option value at times $t \leq T$. To be risk free at time t, a portfolio must have no exposure to $dS_{1,t}$ or $dS_{2,t}$. Suppose the option price is a function $f(S_{1,t}, S_{2,t}, t)$. Let Δ_1 and Δ_2 be the short positions of S_1 and S_2 in the portfolio, in addition to one long remorse option. Find formulas for Δ_1 and Δ_2 in terms of $f(s_1, s_2, t)$. Part of this is figuring out what Ito's lemma says for functions of two independent processes $S_{1,t}$ and $S_{2,t}$. Explain your reasoning.

(c) Write the partial differential equation satisfied by $f(s_1, s_2, t)$. What final conditions does f satisfy?

4. Consider a *power law* option that pays S_T^p at time T.

   (a) Find a formula for $f(s,t) = E_{s,t}\left[S_T^p\right]$ using the explicit formula for the solution of $dS_t = rS_t dt + \sigma S_t dW_t$ starting with $S_t = s$ (sorry for the conflict of notation, note, no discounting in the expectation).

   (b) Write the PDE f is supposed to satisfy, calculate the appropriate partial derivatives of your answer to part (a), and check that these partial derivatives make the equation satisfied.

5. Consider the function $f(w,t) = E_{w,t}\left[e^{\int_t^T W_s ds}\right]$. Repeat the steps of question 4 for this case. Calculate the expectation directly[3], write the PDE f is supposed to satisfy, verify that it does.

6. Suppose discount factors $B(0,t)$ are given for the present value of payments at time t. Suppose we receive payment $V(X_t)dt$ at time t for the time interval $(t, t+dt)$. Write the PDE you would use to compute the present discounted expected value of all payments between time 0 and time T. Formulate an appropriate value function that does not ask for the unknown future yield curve structure $B(t,T)$.

[3] The integrand is Gaussian.