Derivative Securities, Courant Institute, Fall 2010

http://www.math.nyu.edu/faculty/goodman/teaching/DerivSec10/index.html

Always check the class bboard on the blackboard site from home.nyu.edu (click on academics, then on Derivative Securities) before doing any work on the assignment.

Assignment 9, due November 10

Corrections: (None yet)

1. This problem asks you to do Monte Carlo simulations of a diffusion process and compare the results to what you get be solving the backward equation. If $dX = a(X)dt + B(X)dW$ is a diffusion process, the forward Euler approximation is

$$X_{k+1} \;=\; X_k \,+\, a(X_k)\delta t \,+\, b(X_k)\delta W_k \;. \qquad (1)$$

It is clear that when $\delta t \to 0$ this says more or less what the Ito differential equation says. Here we use our usual notation $t_k = k\delta t$, $W_k = W_{t_k}$, $\delta W_k = W_{k+1} - W_k$. The increments δW_k are independent Gaussian random variables with mean zero and variance δt. We will apply this in the case X_t is a stock price, not the log price with a fixed risk neutral return rate r and a price dependent volatility $\sigma(s)$ as we did in assignment 8. The codes that do most of this are `path.cpp`, `randnv.cpp` and `randMain.cpp`. Download them and see that they work together. Modify the codes to treat the case of constant volatility and verify that they correctly price a vanilla European call or put. Make sure to choose parameters that are at the same time realistic for computation and real tests of the code (i.e. not too far in or out of the money, not too close or too far from expiration, etc.). Next, put in the state dependent volatility you used in assignment 8 and see that the expected value from Monte Carlo agrees with the value computed using the Euler method applied to the backward equation. Note that you have to take a very large number of sample paths to get an accurate estimate from Monte Carlo.

2. Consider a model with three credit ratings, A, B, and D, or default. Suppose the only non-zero transition intensities are λ_1 for $A \to B$ transitions and λ_2 for $B \to D$ transitions.

   (a) Write the 3×3 matrix, L, of transition rates.

   (b) Suppose the rating at time 0 is A. Write a formula for $u_1(t)$, the probability that $X(t) = A$.

   (c) Still assuming $X(0) = A$, and using the result of part (b), write a formula for $u_2(t) = \Pr(X(t)\ B)$.

(d) Let T be the time of the $B \to D$ transition. Find a formula for its probability density, $f(t)$.

(e) Assuming that this is the risk neutral model, write a formula for $s_A(T)$, the term structure of credit spread for debt rated A that is due at time T. Assume that this is for a zero coupon bond that matures at time T.

(f) Describe these results qualitatively. What is the behavior of the A bond credit spread for very short times? How does this compare to the credit spread in the simple exponential default model? Does the credit spread increase or decrease with time?

3. In the finite state space Markov chain model, suppose you get a payout V the first time you touch a given state. Hitting times often are called τ. Let $\tau = \min\{t \mid X(t) = n\}$, which is the first time $X(t) = n$. If state n represents default, this would be the default time. If $\tau > T$, you get no payout. Write a system of differential equations similar to the backward equation (7) from the week 9 notes that determines

$$f_i(t) \;=\; E\left[e^{-r(\tau-t)}\mathbf{1}_{\tau\leq T}\right] \; .$$

The random variable $\mathbf{1}_{\tau\leq T}$ takes the value 1 if $\tau \leq t$ and 0 if $\tau > T$. What are the final conditions for f? Use this to write a ordinary differentials that can be solved to determine the present expected value of the continuous time CDS payout structure described on the bottom of page 5 of the week 9 notes.

4. In the binary copula model described in the week 9 notes,

(a) Calculate the correlation coefficient between X_i and X_j, for $j \neq j$ as a function of a.

(b) (harder, do this as time permits) Calculate the approximate distribution of $\sum X_i = \#$ of defaults when n is large using the central limit theorem and other ideas. What is the difference between the case $a = 0$ and $a > 0$?