**Derivative Securities, Fall 2010**
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### Week 1

# 1 Introduction

These notes are supplements to the textbook (Hull, seventh edition) and the lecture notes of Bob Kohn and Steve Allen. These notes are more mathematical, leaving institutional background to the other sources. I mean them mostly as an alternative to bulleted slides that people often lecture from. They are a record of the classes.

The *Derivative Securities* class is an introduction to the quantitative methods used in the *sell side* of the financial services industry. These services include giving their customers access to financial resources such as loans and currencies, and protecting their customers from financial risks such as unknown future interest rates and asset values. These services come in the form of *financial products* such as loans, options, and various ways to buy and sell financial risk.

The official business model[1] of the financial services industry is that the financial services industry itself does not profit from trading, but from the fees they charge for their products. The product provider determines what it costs him or her to create a financial product, then charges the customer slightly more. Competition within the industry keeps fees low, drives people to develop innovative new products, and forces them look for cheaper ways to provide given products. In reality, financial services providers know the markets better than others and have better access to them. They are tempted to use these advantages to profit from trading themselves.

The quantitative side of the industry builds models of financial markets and mathematical analysis of these models to design strategies and estimate risk. The models are not very accurate and make many simplifying assumptions, but the alternative of not using models is worse.

> "All models are wrong. Some models are useful" – George Box

Some of the modeling assumptions concern the way prices change over time. These are the most obvious sources of over-simplification. Other assumptions involve the nature of financial markets and trading and pass without notice. I will refrain from giving long lists of assumptions here, but will point them out as they come up.

[1] The *business model* of a company is the mechanism it uses to make money from the services it provides.

One reason to be aware of your assumptions is that models implicitly assume that certain things cannot happen, which do in fact happen. For example, we treat the time variable, $t$ as continuous when it in fact is discrete, and discrete in a complicated way. let $S(t)$ be the price of a stock at time $t$. We treat this as a continuous function of $t$, while in fact it can jump. A *stop loss* strategy to sell a stock as soon as it "touches" a level $S_0$ can fail if the stock jumps from a value above to a value below. The crash three crashes ago was caused in part because trading was suddenly suspended while prices continued to fall. We will see many more examples of extreme simplification that is motivated by the quote from George Box above, but might more properly inspire:

> "All models are wrong. Some models are dangerous."

## 2 Forward contracts

A *forward contract* is an agreement today to do a transaction at a specified time in the future with specified terms today. A simple forward contract, what is usually meant by the term, is an agreement to buy a specified amount of something at time $T$ for a specified price, $F$. Forward contracts, or simply *forwards*, are struck (entered into) usually without money changing hands, but simply with a "handshake". By convention, "today" is time $t = 0$. Once the *delivery date* is specified, it remains to determine the value of $K$ so that the contract is "worthless" today. This does not mean that the contract has no value, but that the appropriate market price for the contract would be zero. Also, saying that something has no value could mean value $= 0$ or that the value is undefined or not knowable. We mean the former.

The *cash and carry* argument is a simplified theory that determines $F$ under the condition that the forward has zero value at time zero. The theory assumes that the asset has a known price today, which we call $S_0$ or $S(0)$. This often is called the *spot price*. It is the price you pay today if you want the asset today. For example, one can buy jet fuel today for delivery today or one can agree today to buy a specified amount of jet fuel at time $T$ for price $F$. At time $T$ there will be a spot price, which we call $S_T$ or $S(T)$. If I want fuel at time $T$, I can enter into a forward agreement today with a known price $F$, or buy on the spot market at time $T$ at the price $S_T$, which is unknown today.

In the simplest version of the cash and carry argument, one assumes that it is possible to buy the asset today and store it for delivery at time $T$, all for the price today of $S_0$. That is, we can buy the asset today for its spot price but that there is zero *cost of carry* (storage cost, etc.). More realistic versions of the theory assume a cost of carry, often called $q$. One also assumes that it is possible to *sell short* (usually just *short*) the asset today. This means receiving $S_0$ today in return for a negative of the asset. In some markets, this is done by borrowing the asset (jet fuel) from someone who owns it, then selling the borrowed asset. At time $T$, you have to *close* your negative position in the asset by returning the fuel from the person you borrowed it from. If $q = 0$, you did not have to pay rent for the fuel while you were borrowing it.

The final ingredient in the cash and carry reasoning is something about borrowing or lending money. If there is a fixed known interest rate, $r$, you can take a dollar (one unit of currency) at time 0 and repay $e^{rT}$ units of currency at time $T$. Alternatively, you can lend one dollar today and receive $e^{rT}$ dollars at time $T$. The simple theory assumes *borrowing equals lending*, which means that the interest you pay to borrow is the same as the interest you receive when lending. It also neglects *credit risk*, which is the possibility that you might not repay the loan or that the person you lend to might default. Finally, you assume that you can take any amount of loan, positive (borrowing) or negative (lending). It does not have to be an integer number of dollars, or even an integer number of cents.

It is not necessary to assume that $r$ is constant. Instead one can assume that there is a known function $B(0,T)$, which determines the interest on a loan begun today and repaid at time $T$. This is called the *discount bond* price with *zero coupon*. In this scenario, you pay $B(0,T)$ today to receive one dollar at time $T$ (lending), or you receive $B(0,T)$ today and repay one dollar at time $T$ (borrowing). Again, you assume that you can take any amount of loan. This is a *discount bond* because we buy one dollar at the discounted price $B$. It is a *zero coupon* bond because there are no *coupon payments* between today and time $T$.

If there is a constant interest rate, then

$$B(0,T) \;=\; e^{-rT} \; . \tag{1}$$

In real markets, the relation (1) does not hold. Instead there is an effective interest rate that depends on the length of the loan[2]

$$r(0,T) \;=\; \frac{-1}{T}\,\log(\,B(0,T)\,)\; . \tag{2}$$

The graph of $r(0,T)$ as a function of $T$ is called the *yield curve*. For the most part $r(0,T)$ is an increasing function of $T$. If this is not so, the offending part of the yield curve is called *inverted*. The short time limit $\lim_{T\to 0} r(0,T)$ is called the *short rate* or the *overnight rate* (there being no loans for less than overnight). Today, the short rate is ridiculously low, see
`http://www.ustreas.gov/offices/domestic-finance/debt-management/interest-rate/yield.shtml`
The one month rate is .13%. The one year rate is .25%. This means that you lend \$1000 for one year you get \$2.50 in interest[3]. We also have $r(0,10\text{ yr}) = 2.61\%$ and $r(0,30\text{ yr}) = 3.67\%$. These rates are so low that they indicate that something is seriously wrong with our economy. To be fair, a yield curve based on US Treasuries. The yield curve based on LIBOR is not quite so low. Last year,

[2]This is an over-simplification in many ways, one of which is the fact that there exist market prices for zero coupon bonds only for small $T$. Later classes will discuss more realistic ways to construct a yield curve.

[3]To get an idea how small the interest rate is, suppose you had an extra \$1000 and were planning a vacation to Hawaii. You have a choice between going now or postponing the vacation for a year. The US government will pay you \$2.50 to postpone for a year.

one of the questions on the final was the ten year Treasury rate. The correct answer for today is above.

I give the cash and carry argument from the point of view of the financial services provider. The airline company wants to *go long* on a forward contract and contacts the provider to *take the other side* (*go short* the same contract). The provider asks: "What forward price, $F$, can I provide at zero risk to myself without making a profit or taking a loss?" The answer is the *forward price*, given in the two versions

$$F = \begin{cases} e^{rT} S_0 & \text{constant rate} \\ \frac{1}{B(0,T)} S_0 & \text{non constant rate} \end{cases} \tag{3}$$

I explain the second, which includes the first. The provider borrows $S_0$ today and uses the money today to buy the asset. Then he/she waits until time $T$, storing the asset at zero cost. At time $T$, the customer pays $F = (1/B)S_0$ and receives the jet fuel. The provider uses the money to repay the loan. Note that this strategy is risk free in that it does not depend on $S_T$. If $F < S_T$, people will comment on the canny management of the airline and the stupidity of the financial services industry. If $F > S_T$, the same people will chide the airline for having "gone to bed" with "Wall Street types", instead of buying the the fuel on the spot market at time $T$.

What the provider has done is to *replicate* the forward contract. The airline might not have been able to do so directly because it has different access to financial markets. Replication means that in principle it is unnecessary for the forward contract to exist because it can be replicated at a known cost. Replication is a very good way to think about prices of options too.

Suppose that there were a public *exchange* where forwards could be bought or sold at a publicly known market determined price. Suppose that $T$ is the delivery time and $K$ is the price to be paid at that time for the asset. We say $K$ instead of $F$ because $K$ does not change with time. This $K$ is called the *strike price* because it is the number determined at the time the forward was struck. Let $S_t$ be the spot price of the asset time $t$ between today ant $T$. Let $P_t$ be the market price of the forward contract at time $t$. It does not matter for this paragraph, but we may suppose that the forward contract is created today ($t = 0$) with $K = F_0 = (1/B(0,T))S_0$. That does not mean that $K$ will be the forward price at time $t$. In general, $K \neq F_t = (1/B(t,T))S_t$. We want to say something about the market price, $P_t$, of the forward in that case. The forward price $F_t$ is known at time $t$ but not before. Both of the factors $B(t,T)$ and $S_t$ become known at time $t$ but not before.

There is an *arbitrage argument* based on replication that determines $P_t$ at time $t$. An *arbitrage* is a way to make a positive non-zero amount of money with zero risk (more precise definition later). An arbitrage in a market indicates some kind of mispricing or inconsistent prices. For example, if it were possible to buy a Dollar for one Euro, a Euro for two Yen, and three Yen for one Dollars, then one could do a combination of these trades to make a profit with zero risk. The principle of no arbitrage states that this cannot happen in a real market. If it would happen, there would be so much demand for the currencies that the prices would move into *balance*, the no arbitrage situation. More on this later.

In the present situation, suppose the price of the forward contract is $P_t$. At time $t$, we can consider the as an obligation to pay $F_t$ plus an obligation to pay $D_t = K - F_t$ (without saying whether this is positive or negative). The obligation to pay $F_t$ can be replicated without risk at zero cost. The obligation to pay $D_t$ thus may be thought of as a loan. The price of this loan at time $t$ should be $B(t,T)D_t$. The arbitrage argument says that this must be equal to the market price

$$P_t \;=\; B(t,T)D_t \;=\; B(t,T)\left(K - F_t\right) \;=\; B(t,T)K - S_t \; . \tag{4}$$

If $P_t$ is not given by (4), then there is an arbitrage. The trades involved are easy to figure out and are described in the references.

# 3 Futures markets and settlement

Futures contracts are something like exchange traded forwards. An exchange will have a *futures* price at time $t$ for delivery of an asset at time $T > t$. We call this price $G_t$, but that is not standard notation. It might seem that the principle of no arbitrage implies that $G_t = F_t$, but this is not the case because (in addition to the practical difficulties in replication, depending on the asset) of *daily settlement*. Ignoring *margin requirements* for a while, a trader enters a futures market (takes a *position* simply by agreeing to go *long* (agree to take delivery) or *short* (agree to deliver) the asset at time $T$ for the price $G_t$. This price is determined by supply and demand as described in Hull. No money changes hands at this moment.

By the end of the day, which we call time $t_1$, the futures price probably has changed to $G_{t_1} \neq G_t$. The trader with the long position will pay the difference, $G_{t_1} \neq G_t$. Because of settlement, the trader may have to come up with extra cash or find something do do with extra cash at intermediate times. A longer arbitrage argument shows that if interest rates are known, there is an arbitrage opportunity unless $G_t = F_t$. If future interest rates are not known, this argument does not apply.

# 4 Basic options

An *option* is the right to do something but not the requirement to do it. The *holder* of an option must decide whether the *exercise* the option or not. An options contract may have many features, but one with the fewest features is called *vanilla*, or *plain vanilla*. More features make an option *exotic*. A *call* option to buy an asset for a specified *strike price* (or simply *strike* $K$. A *European style* call is the right to buy at a fixed *exercise date*, $T$. An *American style* call is the right to buy at price $K$ at any time up to $T$. A *put* option is the right to sell the asset at price $K$. There are European and American style puts.

Consider a vanilla European put option with strike $K$ and expiration time $T$. This is an option to sell a share of stock at time $T$ for the price $K$. The stock is called the *underlier*. The option is a *derivative* because its value is derived from the price of the underlier. Each European style option has a *payout* function, $V(S_T)$, which is the value of the option to the holder at the time $T$ as a function of the price of the underlier at time $T$. The put is worthless, $V(S_T) = 0$, if $S_T \geq K$. This is because the holder can receive more for a share of the underlier on the spot market than by exercising the option. For that reason, the holder chooses not to exercise the option. In this case, the option expires *out of the money*.

If $S_T < K$, the option has expired *in the money*. In this case, the holder chooses to exercise the option and sell a share of underlying stock for price $K$. If he or she does not own a share, he or she first buys that share on the spot market for $S_T$. Thus, he or she spends $S_T$ and receives $K > S_T$, which amounts to a profit of $V(S_T) = K - S_T$ if $S_T > K$. Almost all commonly traded options are *settled in cash*. This means that the writer of the option simply pays the holder of the option $V(S_T)$ in cast at time $S_T$. The holder of the option is called *long* the option and receives payout $V(S_T)$. The writer is called *short* the option and "receives" $-V(S_T)$. Receiving $-V$ is the same as paying $V$.

The graph of the payout function is the *payout diagram*. You can find many of these in Hull. Each European style option is characterized by its payout function/diagram. A European call has payout $V = 0$ for $S_T \leq K$ and $V(S_T) = S_T - K$ if $S_T > K$. This also is written $V(S_T) = \max(0, S_T - K)$. That expression indicated the optionality. There are two possible payouts. If you exercise, you get $S_T - K$ because you sell the share for $S_T$ and buy for $K$. If you choose not to exercise, you get zero. Because it is an option, you make the decision that gives the larger of these possible payouts. The call payout function also is written as $(S_T - K)_+$, which means the *positive part* of $S_T - K$.

There are many exotic European style options. A *digital option* pays one dollar if $K_1 \leq S_t \leq K_2$. A *capped call* is a vanilla call with a maximum payout $C$. Its payout function is $V(S_T) = \min((S_T - K)_+, C)$. A capped call is safer for the option writer because a vanilla call is a potentially unbounded liability. A *log contract* is a European style option with payout $V(S_T) = \log(S_T)$ (or some multiple of this). Log contracts are used to hedge variance swaps, as you may learn in the next class Continuous Time Finance. Associated to the log contract financial instrument there is an actual legal contract written by lawyers. It may be the only legal contract that includes the mathematical definition of the log function and the number $e$.

Vanilla options are traded on public exchanges (see Hull for details) and have market prices. There are European and American style options on individual equities (stocks), equity indices such as the S&P 500, currency exchange rates, interest rates, etc. Options that expire soon (within a few months) are *short dated*. Options with longer expiration (years or, in rare cased, decades) are *long dated*. Options that are likely to have $V(S_T) = 0$ are *out of the money*, or *deep* out of the money. If there is a reasonable but not overwhelming probability of a positive payout, the option is *near the money*. What is near the money depends on $S_t$, $K$, and $T-t$ (the time to expiry). An asset price has more change to move into the money in a longer time.

A major asset such as the S&P500 index or Dollar/Euro exchange rate has many publicly traded options, puts and calls, with a range of expiries and strikes. These options prices contain information about the "market" opinion of the future behavior of the asset. If the call is expensive, that may mean that the market "expects" the asset price to increase, or it may reflect a large uncertainty in the future price. The relation between option price and the market views of future asset prices is one of the main goals of the theory of option pricing. *Implied volatility* refers to the uncertainty in future prices of the underlier implicit in today's option prices.

When looking at option prices, it is important to know that many options, particularly long dated options or options far from the money are *thinly traded*, meaning that there are few transactions involving that option. In this case, the published price may not be so close to the price the next trader would get for the option.

Suppose a vanilla European put (with some strike and expiry) has price $P_t$ when $t < T$. Suppose that $T-t$ is small enough that we can ignore difference between a dollar at time $t$ and a dollar at time $T$. An investor considering buying the put has a total P&L structure (for *profit and loss*) that includes the option price and its payout. This is simply the option payout with the purchase price subtracted, $(K - S_T)_+ - P_t$ in this case. If the option finishes out of the money, the total $P\&L$ is the purchase price of the option. Call prices are often called $C_t$. An investor who writes (goes short, or *shorts*) a call option has P&L equal to $(S_T - K)_+ - C_t$. Homework 1 discusses a case of a somewhat out of the money call. In most cases, the writer makes a profit of $C_t$, but if the underlier goes up beyond $K + C_t$ (unlikely but possible), the writer takes a loss.

# 5 The lognormal distribution

A random variable $Y$ has a *lognormal* distribution if $\log(Y) = X$ is normal. The probability density for a normal with mean $\mu$ and variance $\sigma^2$ is[4]

$$f(x) \;=\; \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)} \; .$$

This is written $X \sim \mathcal{N}(\mu, \sigma^2)$. The symbol $\sim$ means "has the same probability distribution as", so $X_1 \sim X_2$ means that $X_1$ and $X_2$ have the same probability density. It is not hard to write a formula for $g(y)$, the probability density of $Y = e^X$, but is usually is easier not to. For example expected value of $Y$ is

$$E\left[Y\right] \;=\; \int_0^\infty y\, g(y)\, dy \; ,$$

[4]If you are not familiar with this fact, you may not have the background for this class. It would be good to review a basic probability book such as that by Sheldon Ross. If you just need a review, the Schaum's outline on probability is excellent.

but it is simpler to calculate it using $f(x)$ as

$$E[Y] \;=\; = \; E\left[e^X\right] \;=\; \int_{\infty}^{\infty} e^x f(x)\,dx \;=\; \frac{1}{\sqrt{2\pi\sigma^2}}\int_{\infty}^{\infty} \exp\left(x - \tfrac{1}{2\sigma^2}(x-\mu)^2\right)\,dx\;.$$

It is even easier if you use the substitution $x = \mu + \sigma z$, so that $\frac{1}{\sigma^2}(x-\mu)^2 = z^2$.

It is common to model a future unknown asset price as a lognormal:

$$S_T \;=\; S_0 e^X \quad, \qquad \text{where } X \sim \mathcal{N}(\mu, \sigma^2). \tag{5}$$

*Warning:* do not memorize this formula. We will soon change the meaning of $\mu$ and $\sigma$. The $\mu$ in (5) is *not* the expected return of the asset and $\sigma$ is not the volatility. The model (5) has several useful important features. One is that $S_T > 0$ always. If you would model $S_T$ as a normal, it always would have some probability of being negative. Another is that if $\mu = 0$ the probability of going up by a given percentage $(S_T/S_0)$ is the same as the probability of going down the same percentage. The normal, $X$, is symmetric in the sense of going up or down by the same amount. For example, the probability of going from $S_0 = 100$ to $S_T \approx 50$ is the same as the probability of going from 100 to $S_T \approx 200$. Going down 50 is as likely as going up 100. A related feature is that the lognormal distribution can be very skewed. If $\sigma$ is large, it is very unlikely that $S_T \geq E[S_T]$. By contrast for any parameter values $\Pr(X > E[X]) = .5$.

The lognormal is "derived" from the geometric Brownian motion model of asset prices used by Black and Scholes, but this is not a very deep fact, because that model essentially assumes the main features of the lognormal.

The *return* of an investment is

$$\text{return} \;=\; \frac{\text{Value at time } T - \text{initial value}}{\text{initial value}}\;.$$

If you buy one unit of an asset, the return is $(S_T - S_0)/S_0$ In the lognormal model, the return distribution is a lognormal. Now consider an investment that consists of the underlier and a short position in some vanilla European call on the underlier that expires at time $T$. This *portfolio* has a return distribution that depends on the distribution of $S_T$. The basic theory of investments is based on the belief that an investor or fund manager should decide his or her investment portfolio by comparing the return distributions of different portfolios.

# 6 Monte Carlo

*Monte Carlo* means using computer generated random numbers to evaluate some quantity. It is a fundamental computational method in finance, particularly when quantities are defined in terms of probability models. Suppose $Y$ is a random variable with a given distribution, and we want to estimate $E[Y]$. Suppose further that it is possible to generate $n$ independent *samples* of the random variable $Y$. This means that you can make independent random variables $Y_k$

for $k = 1, \ldots, n$ with $Y_k \sim Y$. The law of large numbers in probability states that if $n$ is large then

$$E[Y] \;\approx\; \frac{1}{n}\sum_{k=1}^{n} Y_k \;, \tag{6}$$

which is to say that the sample mean is approximately equal to the population mean. The simplest Monte Carlo method is to use the computer to evaluate the right side of (6). For large $n$, this gives an accurate estimate of $E[Y]$. Monte Carlo always gives an approximate answer, it never is exact (well, almost never).

You can use Monte Carlo to estimate the probability density of $Y$. For this you make a large number of samples $Y_k$ and make a histogram. The height of the bars is an estimate of the probability density.