Derivative Securities, Fall 2010
Mathematics in Finance Program
Courant Institute of Mathematical Sciences, NYU
Jonathan Goodman
http://www.math.nyu.edu/faculty/goodman

Week 2

1 Introduction

This class begins the discussion of option pricing. We use an abstract model to explain the related concepts of *complete markets*, *arbitrage pricing*, and *risk neutral probability*. We will stick mostly with *one period* models, leaving multi-period models for the end. We start with the one period binary model that is the basis of the Cox Ross Rubenstein (*CRR*) *binomial tree* pricing model. We then give a more general version of the same argument and explain how the binary model fits into it.

You may find this week's notes particularly repetitive. As an excuse I can only say that I have taught this material many times in many ways, and have fielded many questions. Each repetition is designed to answer one set of questions that repeatedly have been asked.

2 One period binary model

This model considers a single *risky* asset and an option on that asset. The asset price today is S_0. The price "tomorrow" is S_T. The price today is known. The price tomorrow is either $S_T = uS_0$ or $S_T = dS_0$, with $u > d > 0$. The numbers u (for "up") and d (for "down") are known today. The asset is called risky because S_T is not known today. It is called *binary* because there are only two possible *outcomes* tomorrow. The model also assumes there is a risk free asset, which I often call *cash*. Using the bond notation from last week, one unit of cash is worth B today and 1 tomorrow.

In this model, the option will make a payout tomorrow, paying $V(S_T) = V_u$ if $S_T = uS_0$ and $V(S_T) = V_d$ if $S_T = dS_0$. It is of course possible that V_u or V_d is negative, particularly if we are talking about a short position in an option. If the option is a (long position in a) vanilla put, then $V(S_T) = (K - S_T)_+$. This is interesting only if $dS_0 < K < uS_0$. Otherwise there is no difference between the option and the underlier.

Said slightly differently, the model assumes an "economy" consisting of three instruments: a risky asset, a risk free asset, and a contingent claim on the risky asset. *Contingent claim* means that the value of the instrument is contingent on the value of the underlying risk asset. The model has two "states of the world", one in which $S_T = uS_0$ and one in which $S_T = dS_0$. The investor may buy any amount, positive or negative, of the risky or risk free asset at the prices S_0 and B respectively.

There are (at least) three ways to give the option pricing argument for the binary one period model: the *replication* argument, the *arbitrage* argument, and the *hedging* argument. These are not really different. They use the same algebra and yield the same option price.

The replication argument is similar to the one we gave last week for pricing a forward contract. In the present case, we *replicate* the option using a portfolio consisting only of stock and cash. The portfolio is chosen so that its value tomorrow is the same as the value of the contingent claim, in every state of the world tomorrow. If the value tomorrow is the same, then the value today also must be the same. The two are identical from a financial point of view.

The replicating portfolio consists of Δ units of the stock and M units of cash. The value of this portfolio today is

$$\Pi_0 \;=\; \Delta S_0 \,+\, MB\;. \tag{1}$$

The same portfolio tomorrow is worth

$$\Pi_T \;=\; \Delta S_T \,+\, M\;. \tag{2}$$

The portfolio replicates the contingent claim (option) if

$$V(S_T) \;=\; \Pi_T \tag{3}$$

in every state of the world tomorrow. This leads to two equations, one for each possible state:

$$\left.\begin{array}{rcl} V_u &=& uS_0\Delta \,+\, M \\ V_d &=& dS_0\Delta \,+\, M\;. \end{array}\right\} \tag{4}$$

The unknowns are the portfolio *weights* Δ and M. We solve by subtracting the second equation from the first,

$$V_u \,-\, V_d \;=\; (u-d)S_0\Delta\;,$$

then solving for Δ,

$$\Delta \;=\; \frac{V_u - V_d}{(u-d)S_0} \;=\; \frac{V_u - V_d}{uS_0 - dS_0}\;. \tag{5}$$

In a similar way, we find

$$M \;=\; \frac{uV_d - dV_u}{u-d}\;. \tag{6}$$

Substituting (5) and (6) back into (1) gives

$$\Pi_0 \;=\; \frac{1-Bd}{u-d}V_u \,+\, \frac{Bu-1}{u-d}V_d\;. \tag{7}$$

Since Π replicates the option tomorrow, it must replicate the option today. That means that the price (7) of the portfolio today also is the price of the option today.

This may be restated as an arbitrage argument. Let f be the price of the option. We argue that unless $f = \Pi_0$, an investor can make an arbitrarily large profit with no risk. Suppose, for example, that $f < \Pi_0$. In that case, we could short the portfolio, that is, short Δ shares of stock and borrow MB in cash. In total, this gives us Π_0 cash. We spend f of this to buy the option and put the remainder away. At time T, we get either V_u or V_d from the option, which we use to repay the loan and buy back the Δ units of stock. In either the u or the d state, $V(S_T)$ is exactly enough money to do this, as (4) states. We are left with $\Pi_0 - f > 0$ cash no matter what happened.

The *no arbitrage argument* is that if $f < \Pi_0$, so many people would do the arbitrage that the prices would adjust to remove it. There would be so much demand for the option that its price would increase. So many people trying to sell the stock would send its price lower. Eventually (very quickly in live markets), the arbitrage opportunity would disappear.

If $f > \Pi_0$, the arbitrage is reversed. The trader would short the option and receive f, spend Π_0 of that f to buy Δ shares of stock, make a risk free loan of size MB, and put the remaining $f - \Pi_0 > 0$ away. Tomorrow, she sells the stock, does the option payout, and still has $f - \Pi_0 > 0$ with zero risk.

The third (and last) argument for the pricing formula (7) is the hedging argument. It is a variation on the replication argument, which says that if a portfolio looks like the option, it is the option. The hedging argument says that if a portfolio looks like cash, it is cash. More precisely, if Π_T is known today independent of the state of the world tomorrow, then Π is a risk free asset and grows at the risk free rate:

$$(\, \Pi_T, u \; = \; \Pi_T, d \,) \;\Longrightarrow\; (\, \Pi_0 \; = \; B\Pi_T \,) \; . \tag{8}$$

Here, the portfolio Π will be a *risk free hedge* that is long one and short Δ shares of stock. The hypothesis of (8) for this portfolio is

$$V_u - \Delta u S_0 \; = \; V_d - \Delta d S_0 \; . \tag{9}$$

This of course leads to the same hedge ratio (5). The conclusion of (8) becomes

$$f - \Delta S_0 \; = \; B\left(V_u - \Delta u S_0)\right) \; = \; B\left(V_d - \Delta d S_0\right) \; .$$

Some algebra shows that the f here is the same as the right side of (7).

The algebra of risk free hedging is simple and intuitive. We want $\delta\Pi_T = 0$, where $\delta\Pi_T$ is the difference between the Π_T values in the up and down states. With $\Pi_T = V(S_T) - \Delta S_T$, this is simply $\delta V - \Delta \delta S_T = 0$, which is one of the forms of (5). The Black Scholes theory asks us to take the limit of the formula $\Delta = \frac{\delta V}{\delta S}$ as $\delta S \to 0$. This is the Black Scholes hedging formula $\Delta = \frac{\partial V}{\partial S}$.

We can test the pricing formula (7) in two simple cases, cash and the stock. Cash is the case $V_u = V_d = 1$, which is to say that the supposedly "contingent" claim actually pays one dollar no matter what happens. In this case, the option actually is a zero coupon discount bond and its price today should be B. An easy check shows that (7) indeed gives $\Pi_0 = B$ in that case. Another case is

$V(s) = s$, which is $V_u = uS_0$ and $V_d = dS_0$. Plugging these values into (7) and simplifying gives $\Pi_0 = S_0$, as it should. This fits with the replication argument. If $V(S_T) = S_T$ then V is identical to S_T as a financial instrument. This implies that the option is identical to the asset also today, which is $\Pi_0 = S_0$, they have the same price.

All this has a fatal flaw if either of the coefficients on the right of (7) is negative. The denominators are positive, so this is a statement about the numerators. Suppose, for example, that $1 - Bd < 0$. This means that $d > 1/B$, which in turn implies that the risky asset makes money relative to the risk free bond no matter what. An arbitrage argument would say that an investor today could borrow S_0 and buy one share of stock. Tomorrow, she or he sells the stock. If $S_T = dS_0$, then she or he receives dS_0, repays the loan by paying S_0/B, and keeps the remaining $(d - (1/B))/S_0 > 0$. If $S_T - uS_0 > dS_0$, she or he keeps even more money. In either case, she or he used zero money (total) at today and has a guaranteed positive amount of money tomorrow, which is the definition of arbitrage. You should think through the arbitrage that is possible if the second coefficient is negative. In that case, the risk free asset outperforms the risky asset in every state of the world tomorrow. The arbitrage strategy is to short the stock today and lend S_0 today at the risk free rate. Tomorrow, you have more than enough money to buy back the stock, even if $S_T = uS_0$.

Finally we come to risk neutral probabilities. If f is the option price today and $f = \Pi_0$, then (7) may be written

$$f \;=\; A_u V_u \;+\; A_d V_d \;\;, \quad \text{with } A_u \;=\; \frac{1 - Bd}{u - d} \;\;, \quad A_d \;=\; \frac{Bu - 1}{u - d} \;. \tag{10}$$

If there is no arbitrage, $A_u \geq 0$ and $A_d \geq 0$. A simple calculation shows that

$$A_u \;+\; A_d \;=\; B \;.$$

This implies that p_u and p_d defined by

$$p_u \;=\; \frac{1}{B} A_u \;\;, \quad p_d \;=\; \frac{1}{B} A_d \tag{11}$$

may be thought of as probabilities ($p_u + p_d = 1$, $p_u > 0$, $p_d > 0$). The option pricing formula (7) may be written as

$$f \;=\; B\left(p_u V_u \;+\; p_d V_d\right) \;=\; B E_{RN}\left[V(S_T)\right] \;. \tag{12}$$

In the last expression, $E[\cdots]$ refers to expected value, and $E_{RN}[\cdots]$ refers to expected value using *risk neutral probabilities* p_u and p_d (the risk neutral probabilities that $S_T = uS_0$ or $S_T = dS_0$ respectively). We say that $E_{RN}\left[V(S_T)\right]$ is the risk neutral expected value of the option tomorrow. To get the value today of that expected value tomorrow, we multiply by the discount factor B. If Q is an amount of money tomorrow, the value of that money today (the price of that money today) is BQ. This is called the *present value*, or the *discounted value* of Q. Thus (12) represents the present value of the expected value of the payout in the risk neutral probabilities given by (11).

3 Risk neutral measure

To understand the term *risk neutral measure* you need to know something about the basic philosophy of investments. This is the idea that risky investments involve a tradeoff between risk and expected value. Suppose you have a dollar and are choosing between investments X and Y. If $E[X] > E[Y]$ one would tend to prefer X to Y. If $\text{var}(X) > \text{var}(Y)$, one might prefer Y to X. If $z = E[X]$, then the "rational"[1] investor must prefer z to X because z has the same expected value but no risk. This is *risk aversion.* A risk averse investor would pay z for a payout of z, but would pay less than z for an unknown payout X with expected value equal to z. Different investors have different levels of risk aversion. It is impossible to say exactly what price a risk averse investor would put on X, only that it is less than $E[X]$.

An investor who is not risk averse is *risk neutral.* Such an investor chooses investments depending only in their discounted expected values. If f is the price today of a payout $V(S_T)$ tomorrow, then a formula $f = BE_{RN}[V(S_T)]$ states that the price f is *as if* $V(S_T)$ were valued by a risk neutral investor, if the probabilities of the various outcomes were given by the "risk neutral measure". However, real investors are not risk neutral and $E[V(S_T)] \neq E_{RN}[V(S_T)]$.

Two normal risk averse investors in principle would have different prices that they would be willing to pay today to receive $V(S_T)$ tomorrow, just like they would have different prices they would be willing to pay for a stock or anything else. Economists say that in a world where people value a specific good (item to be owned) in different ways, the price of that good is determined in open markets by the law of supply and demand. If that good is a future payout such as $V(S_T)$, the price today would be determined by this law of supply and demand. Both the supply and the demand would be determined by the levels of risk aversion of people in the marketplace.

But the option pricing theory of the previous section says the opposite. People's risk preferences do not enter into the pricing formula (12). You do not need to know about risk preferences to determine the price of an option. Every rational person would arrive at the same price, regardless of her or his personal level of risk aversion. This is one of the reasons risk neutral pricing was so attractive to people dealing in derivatives. It gave them a rational way to figure out not only how much a put or a call might be worth in the market without knowing anything about the participants in the market. It also gave a way to protect derivatives providers in the financial services industry from market risks associated with derivatives.

We end with a piece of terminology. The actual probabilities of u and d do not enter into (12) but they do exist. Actual price probabilities are called the *P measure*[2]. The actual expected payout of an option is the expected value in the P measure. The risk neutral probabilities in (11) form the *Q measure.*

[1] Real people break this rule all the time. For example, a lottery ticket has a negative expected return and lots of variance. Not everyone lives by the axioms of micro-economics.

[2] A *measure* is a mathematical way of specifying probabilities. We discuss measures just a little in the Derivative Securities class. The Stochastic Calculus class probably has more.

People who do derivatives pricing using replication and hedging arguments, i.e. most of the quants on the sell side, will tell you they work with the Q measure. People on the buy side doing asset allocation etc. who use statistical methods to forecast market movements will tell you they use the P measure.

4 General discrete one period model

This section and the next are not strictly necessary for what follows. I include them partly because the math is so nice. We will use a somewhat abstract form of linear algebra. You can review this kind of linear algebra in the book by Lax, for example.

The abstract discrete one period is a more general version of the binary model above. In the general model there are M possible states of the world "tomorrow", which we label as $j = 1, \ldots, M$. There are N instruments (or assets)in the marketplace today, labelled as $i = 1, \ldots, N$. The market price of instrument i today is f_i. The market value of instrument i tomorrow in state of the world j is V_{ij}. A one period model is determined by the the N numbers f_i, and the $N \times M$ matrix V with entries V_{ij}. We assume these numbers are all known today.

A portfolio is a a set of N *weights* for the N assets. The amount of asset i is w_i. These weights may be thought of as the components of a vector $w \in \mathbb{R}^N$. The portfolio determined by w has w_i units of asset i. As always, there is no requirement that the weights should be integers or positive. Today, the portfolio has value

$$\Pi_0 \;=\; \Pi_0(w) \;=\; \sum_{i=1}^{N} w_i f_i \; . \tag{13}$$

Tomorrow, if the world is in state j, the portfolio has value

$$\Pi_{T,j} \;=\; \Pi_{T,j}(w) \;=\; \sum_{i=1}^{N} w_i V_{ij} \; . \tag{14}$$

The numbers $\Pi_{T,j}$ form the components of a vector $\Pi_T \in \mathbb{R}^M$. The set of all possible portfolio value functions $\Pi_T \in \mathbb{R}^M$ forms a subspace[3] $\mathbf{P} \subseteq \mathbb{R}^M$. At this point, it may be that $\mathbf{P} = \mathbb{R}^M$, or $\mathbf{P}$ may be a proper subspace of $\mathbb{R}^M$ (particularly if $M > N$).

We say that V has a *positive investment* if there is some i so that $V_{ij} \geq 0$ for all j and $V_{ij} > 0$ for some j. For example, a risk free investment has the same positive value in state of the world, j. It is not necessary that V have a risk free instrument. It is even not necessary that instrument i have a positive value for every outcome tomorrow, but it should have a positive outcome for at least one possible outcome.

[3] Recall the definition of subspace – closed under addition and multiplication by scalars. The set of all portfolio value functions has these properties.

It also is possible that there is more than one set of portfolio weights that produces the same payout function. That is, $\Pi_T(w_1) = \Pi_T(w_2)$ while $w_1 \neq w_2$. This must be the possible if $N > M$. It also would happen if two of the instruments, say i and i', were identical, $V_{ij} = V_{i'j}$ for all j. In fact, replication was finding different ways to get the same payout structure.

An *arbitrage* is a portfolio weight vector so that $\Pi_0 = 0$ and $\Pi_{T,j} \geq 0$ for $j = 1, \ldots, M$, with $\Pi_{T,j} > 0$ for at least one state, j. The one period model determined by V is called *arbitrage free* if no weight vector is an arbitrage. In an arbitrage free model with a positive instrument, two weight vectors that determine the same payout have the same cost. We state that formally as

Claim 1. Suppose a one period model determined by f and V is arbitrage free and has a positive instrument. If $w_1 \in \mathbb{R}^n$ and $w_2 \in \mathbb{R}^N$, are two weight vectors with $\Pi_T(w_1) = \Pi_T(w_2)$, then $\Pi_0(w_1) = \Pi_0(w_2)$.

Proof. This is a proof by contradiction. We suppose the conclusion is false and show that the hypothesis (at least some part of it) also must be false. Thus, assume that $\Pi_0(w_1) \neq \Pi_0(w_2)$ but $\Pi_T(w_1) = \Pi_T(w_2)$. Without loss of generality, we may assume that $\Pi_0(w_1) < \Pi_0(w_2)$. We show that under these conditions, if the model has a positive investment, then it is not arbitrage free. We suppose that the positive investment has a positive cost[4], $f_i > 0$. Consider the investment weight vector $w_3 = w_1 - w_2$. The "cost" today of this portfolio is $x = \Pi_0(w_1) - \Pi_0(w_2) < 0$ by hypothesis. Therefore, a portfolio[5] $w_4 = w_3 - (x/f_i)e_i$ has zero cost today, $\Pi_0(w_4) = 0$. We write $V_i \in \mathbf{P}$ for the payout of instrument i. Tomorrow, the value of the w_4 portfolio is $\Pi_T(w_1) - \Pi_T(w_2) - (x/f_i)V_i = -(x/f_i)V_i$. Since $(x/f_i) < 0$, this is an arbitrage. This completes the proof of the claim.

Claim 1 implies that the cost of a portfolio $\Pi_T \in \mathbf{P}$ is determined by Π_T itself. We write this as $f(\Pi) = \Pi_0(w)$. The claim says that f does not depend on which w we use, as long as $\Pi_T(w) = \Pi_T$. This cost is a linear function of Π_T, so it is given as the inner product with a uniquely determined *cost vector* $A \in \mathbf{P} \subseteq \mathbb{R}^M$. This may be stated less abstractly as saying that there are cost coefficients A_j, for $j = 1, \ldots, M$, so that

$$f(\Pi_T) \;=\; \sum_{j=1}^{M} A_j \Pi_{T,j} \; . \qquad (15)$$

The numbers A_j, or, equivalently, the vector $A \in \mathbb{R}^M$, are determined uniquely by (15) if $\mathbf{P} = \mathbb{R}^M$. In this case, the model is said to be *complete.* There is a unique $A \in \mathbf{P}$ even if the model is incomplete. The main result of this section is

[4]If it has zero cost, it would be an arbitrage already. If it had negative cost, it sure would seem like an arbitrate: someone pays you today to hold an instrument that has a positive value tomorrow. Perhaps the definition of arbitrage should have included negative initial cost, but it didn't. Instead, you could modify the proof below to have negative cash today to "buy" (for a negative price) the positive asset.

[5]The vector e_i is the unit vector in the i direction. It has all components equal to zero except component i equal to 1. As a weigh vector, e_i invests only in instrument i.

Claim 2. In a complete market that is arbitrage free, $A_j \geq 0$ for $j = 1, \ldots, M$.
Proof. Let $\mathbf{L} \subset \mathbb{R}^M$ be the set of portfolios that have zero initial cost: $\Pi_T \in \mathbf{L}$ if $\Pi_T = \Pi_T(w)$ with $\Pi_0(w) = 0$. The relation (15) implies that $\Pi_T \in \mathbf{L}$ if and only if $\langle A, \Pi \rangle = \sum_{j=1}^M A_j \Pi_{T,j} = 0$. Geometrically, this means that the vector A is perpendicular to the hyperplane $\mathbf{L}$. Let $Q \subset \mathbb{R}^M$ be the positive quadrant, which means that $\Pi_T \in Q$ if and only if $\Pi_{T,j} \geq 0$ for all j and $Pi_{T,j} > 0$ for some j. The claim is equivalent to saying $A \in Q$. The geometrical fact, which you can check by picture, is that if $A \notin Q$ than some vector perpendicular to A is in Q. But being perpendicular to A is the same as being in $\mathbf{L}$. A portfolio in $\mathbf{L}$ and also in Q is an arbitrage opportunity: it has zero cost today and positive value tomorrow. This is a proof of Claim 2 by contradiction.

Finally we come to risk neutral pricing in general one period models. Define the normalization parameter $B = \sum_{j=1}^M A_j$. The risk neutral probabilities are $p_j = A_j/B$. These are non-negative numbers with $\sum p_j = 1$. Therefore, we may view these numbers as probabilities. The expectation value with respect to the risk neutral probabilities (the risk neutral measure) is

$$E_{RN}[\Pi_T] \;=\; \sum_{j=1}^M \Pi_{T,j} p_j \;.$$

Since $A_j = Bp_j$, the present price formula (15) may be rewritten in terms of risk neutral expectation as

$$f(\Pi_T) \;=\; BE_{RN}[\Pi_T] \;. \tag{16}$$

In this way, here is a Q measure associated to any complete arbitrage free market.

The risk neutral probabilities p_j and discount factor B must price correctly all of the instruments in the market. For instrument i, this means that

$$f_i \;=\; \sum_{j=1}^M V_{ij} A_j \;=\; B \sum_{j=1}^M V_{ij} p_j \;=\; BE_{RN}[V_i] \;. \tag{17}$$

If the market is complete any payout function tomorrow may be replicated by a portfolio. That means that if Π_j is any payout function contingent on the state of the world tomorrow, there is a portfolio weight vector so that $\Pi = \Pi_T(w)$. If the market is arbitrage free, the price today of any portfolio replicating the payout Π is the same. That was Claim 1. The conclusion is that in a complete arbitrage free market, a set of probabilities and a discount factor that satisfies (17) for all the traded instruments also prices correctly any other payout function. Moreover, there is a unique price for each payout function that is independent of the risk preferences of the investor.

It is possible to extend this discussion to markets that are not complete. In that case, the numbers A_j in (15) are not uniquely determined. One can show (but I do not do it here), that it is possible to choose A_j so that all of them are non-negative. These A_j will satisfy (17) for each of the instruments in the market. But even if the market is arbitrage free, there may be other payout functions whose prices are not determined by (17), the traded instruments. That is, there is more than risk neutral measure that correctly reproduces market prices. These different measures assign different prices to other instruments that cannot be replicated by traded instruments. For that reason, risk neutral pricing is less useful in incomplete markets.

5 The binary model as an example

The binary model is an example of the general theory. It has two instruments, $i = 1$ is cash and $i = 2$ is stock. There are two states of the world tomorrow, $j = 1$ is the down state and $j = 2$ is the up state. The prices today of the instruments are $f_1 = B$ and $f_2 = S_0$. The value of cash tomorrow does not depend on the state of the world tomorrow: $V_{11} = V_{12} = 1$. That is the meaning of *risk free*. The values of the stock tomorrow are $V_{21} = dS_0$ and $V_{22} = uS_0$. Suppose a portfolio has weights w_1 (cash) and w_2 (stock). The value today of the portfolio is $f = w_1 B + w_2 S_0$. If $f = 0$, then $w_1 = -w_2 S_0/B$. (Recognize S_0/B as the forward price of the stock.) The portfolio value tomorrow in the down state is $w_1 + dS_0 w_2 = w_2 S_0 \left(d - \frac{1}{B}\right)$. The portfolio value tomorrow is given by

$$\Pi_T \;=\; w_2 S_0 \begin{pmatrix} \left(d - \frac{1}{B}\right) \\ \\ \left(u - \frac{1}{B}\right) \end{pmatrix} \,.$$

Therefore, the subspace $\mathbf{L} \subset \mathbb{R}^2$ is the one dimensional space spanned by the vector

$$\begin{pmatrix} \left(d - \frac{1}{B}\right) \\ \\ \left(u - \frac{1}{B}\right) \end{pmatrix} \,.$$

The vector A for pricing is supposed to be perpendicular to this. A perpendicular to the column vector $(x, y)^t$ is $(y, -x)^t$. Therefore A must be in the direction of

$$\begin{pmatrix} \left(u - \frac{1}{B}\right) \\ \\ \left(\frac{1}{B} - d\right) \end{pmatrix} \,.$$

If there is no arbitrage, the components of this vector have the same sign. As we saw already, the no arbitrage conditions are $u > \frac{1}{B}$ and $d < \frac{1}{B}$, which is the same as saying that both components are positive. We normalize these components to get a probability vector by first adding them

$$u - \frac{1}{B} + \frac{1}{B} - d = u - d \,,$$

then dividing

$$p_1 \;=\; p_d \;=\; \frac{u - \frac{1}{B}}{u - d} \;\;, \quad p_2 \;=\; p_u \;=\; \frac{\frac{1}{B} - d}{u - d} \;.$$

These are the same risk neutral probabilities we found earlier.

6 Multi period binomial model

The new idea here is *dynamic hedging.* The options dealer replicates (or, equivalently, hedges) an option by repeatedly adjusting a portfolio consisting of the underlier (stock) and cash. In each time period, the hedge ratio Δ (the amount of stock) is chosen to make the total position – option + stock + cash – risk free over the next period. The result is that at the final time T, no matter what path the stock price has taken, the option has been reproduced perfectly. The option price is determined by the cost of replicating it. This option price turns out to be the discounted expected payout in a risk neutral system of probabilities that we construct as part of the solution. That is, as in the one period model, the replication model = the hedging model = the pricing model = risk neutral pricing.

The main components of this strategy were discovered by Black and Scholes in a continuous time model using goemetric Brownian motion and continuous time hedge rebalancing. This section covers a discrete time version of their argument due to Cox Ross and Rubenstein. This *CRR binomial tree* pricing model should be thought of as a teaching method, not a serious method to be used in the real world. That was the intention of CRR.

The CRR binomial tree has n time periods, the periods between $n+1$ times $0 = t_0 < t_1 \cdots < t_n = T$. At this point it is not necessary to know what the times are. That will come next week when we talk about calibration. The stock price at time t_k will be called S_{t_k}. We often will write S_k for the more proper S_{t_k}. Each time period consists of the binary model, which means that the possible values of S_{k+1} are $S_{k+1} = uS_k$ or $S_{k+1} = dS_k$. We assume that the risk free rate is a known constant between all periods. That is, at any time t_k you can receive B or lend B and then repay or receive 1 at time t_{k+1}. As a consequence, you can receive B^2 at time t_k and repay 1 at time t_{k+2}, etc.

We suppose that the initial price, the spot price today, is a known $S_0 = S_{t_0}$. The possible values of S_1 are uS_0 and dS_0. The possible values of S_2 are u^2S_0, udS_0, and d^2S_0. There are two different paths to $S_2 = udS_0$, $S_0 \to (S_1 = uS_0) \to) S_2 = dS_1 = udS_0)$, and $S_0 \to (S_1 = dS_0) \to) S_2 = uS_1 = udS_0)$. The tree is *recombining* in that (*up,down*) and (*down, up*) lead to the same price. In the whole tree up to time n, there are 2^n different possible paths, but only $n+1$ possible values of $S_n = S_T$. The possible values are $S_T = u^n S_0$, $S_T = u^{n-1}dS_0$, $\cdots$, $S_T = d^n S_0$.

Suppose there is an option that have a payout function at time T, $V(S_T)$. We want to determine the price of this option at earlier times. This can be done using a backward induction argument. We will determine a sequence of *value functions* $f_k(S_k)$, starting from $f_n(S_n) = V(S_n)$, then moving to $f_{n-1}(S_{n-1})$, and working backwards to $f_0(S_0)$. This will be done so that $f_k(S_k)$ is the price of the option if S_k is the stock price at time t_k. It may happen that you know S_0 and only want to know the spot option price today, which would be the single number $f_0(S_0)$. Even in that case you have to compute all the intermediate functions f_k. If you have a background in control theory or computer science, you may recognize this as the principle of dynamic programming. If you have a background in probability, you may recognize this as a discrete version of the Kolmogorov backward equation (also called Chapman Kolmogorov).

The actual computation is easy. It is what we did a few sections ago. Suppose we are at time t_{n-1} with spot price at that time equal to S_{n-1}. There are two possible prices going forward to time n, $S_n = uS_{n-1}$ and $S_n = dS_{n-1}$. We know the option price for either of these outcomes. Therefore, the price time t_{n-1} is

$$f_{n-1}(S_{n-1}) \;=\; BE_{RN}[V(S_n)] \;=\; B\left(p_u V(uS_{n-1}) \,+\, p_d V(dS_{n-1})\right) \;,$$

where the risk neutral probabilities p_u and p_d are the same as (11). This determines the n numbers $f_{n-1}(S_{n-1})$ in terms of the $n+1$ known numbers $V_n(S_n)$.

This can be continued to times t_{n-2} and so on. At each time t_k, you assume that the prices at the next time t_{k+1} are known. If the stock price at time t_k is S_k, the two possible values at time t_{k+1} are uS_k and dS_k. Therefore,

$$f_k(S_k) \;=\; BE_{RN}[f_{k+1}(S_{k+1})] \;=\; B\left(p_u f_{k+1}(uS_k) \,+\, p_d f_{k+1}(dS_k)\right) \;. \qquad (18)$$

One can use the binary tree argument to prove (within this model) that $f_k(S_k)$ will be the price of the option at time t_k if S_k is the stock price. The induction goes from $k = n-1$ down to $k = 0$. At each time, the left side of (18) is known, so (18) determines the right side.

The equations (18) and the hedging formula (5) give the hedge at time t_k to be

$$\Delta_k(S_k) \;=\; \frac{f_{k+1}(uS_k) - f_{k+1}(dS_k)}{S_k(u-d)} \;. \qquad (19)$$

This allows the derivatives dealer to replicate the payout $V(S_T)$ through a *dynamic hedging* strategy. At time $t_0 = 0$ the dealer starts with cash $f_0(S_0)$. She or he buys $\Delta_0(S_0)$ shares of stock and holds the rest in cash. This is the *replicating portfolio* at t_0. And it is night and it is morning[6] and we come to time t_1 and learn whether $S_1 = uS_0$ or $S_1 = dS_0$. In either case the replicating portfolio constructed at time t_0 has value equal to $f_1(S_1)$. The homework will ask you to verify this.

At time t_1 you have to *rebalance*, which means to construct the new replicating portfolio for the next period. If $S_1 = uS_0$, the portfolio from the previous period is worth $f_1(uS_0)$. You use part of this to buy $\Delta_1(uS_0)$ shares of the stock at price $S_1 = uS_0$ and hold the rest in case. The replicating portfolio at time t_1 with $S_1 = uS_0$ will have a different Δ: $\Delta_1(uS_0) \neq \Delta_0(S_0)$. This means that at time t_1 you have to buy or sell some stock to rebalance. Then, regardless of whether $S_2 = uS_1$ or $S_2 = dS_1$, your portfolio at time t_2 will be worth $f_2(S_2)$.

As we said earlier, there are two paths to $S_2 = udS_0$. The hedging histories for these two paths are different, but either way the portfolio at time t_2 will be worth $f_2(udS_0)$. You will be asked to check this explicitly in the homework too.

Again, by induction, when you reach the final time t_T, no matter what path you have taken and no matter what the final stock price is, the stock and cash

[6] In the first part of the Bible when the world is being created, we go from day to day in this way.

portfolio will be worth $f_n(S_n) = V(S_n)$. That is to say, no matter what happens, you will have exactly the amount of money required to satisfy the option.