Derivative Securities, Fall 2010
Mathematics in Finance Program
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http://www.math.nyu.edu/faculty/goodman/teaching/DerivSec10/resources.html

Week 4

1 Mathematical preliminaries

If X and Y are random variables, then $X \sim Y$ will mean that X and Y have the same probability distribution. In other words, $X \sim Y$ if and only if $E[V(X)] = E[V(X)]$ for any function V. We also write $X \sim f$ if f is the probability density of X. The normal density with mean m and variance v is written $\mathcal{N}(m, v)$, so $X \sim \mathcal{N}(m, v)$ means that X is a normal random variable with those parameters. We already used the fact that if $Z \sim \mathcal{N}(0, 1)$, then $m + \sqrt{v}Z \sim \mathcal{N}(m, v)$. Taking $V(x) = e^x$, we get $E[e^X] = E[e^{m+\sqrt{v}Z}]$, so

$$\frac{1}{\sqrt{2\pi v}} \int_{-\infty}^{\infty} e^x \, e^{\frac{(x-m)^2}{2v}} \, dx \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{mz+\sqrt{v}z} \, e^{-z^2/2} \, dz \; .$$

You can understand this identity simply as using the change of variables $x = m + \sqrt{v}z$, $dx = \sqrt{v}dz$ in the integral. The same change of variables gives

$$\frac{1}{\sqrt{2\pi v}} \int_{-\infty}^{a} e^x \, e^{\frac{(x-m)^2}{2v}} \, dx \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{a-m}{\sqrt{v}}} e^{mz+\sqrt{v}z} \, e^{-z^2/2} \, dz \; .$$

The upper limit on the right comes from $x = a \Longleftrightarrow z = \frac{a-m}{\sqrt{v}}$.

The relation $X \sim Y$ does not imply that X and Y have the same value. For example, if $Z \sim \mathcal{N}(0, 1)$, then $-Z \sim Z$. If Z_k are independent with $Z_k \sim \mathcal{N}(0, 1)$, then $Z_1 + Z_2 \sim \sqrt{2}Z_3$.

If $X \sim f$, then the cumulative distribution function, or *CDF*, is

$$F(x) \;=\; \Pr(X \le x) \;=\; \int_{-\infty}^{x} f(x') \, dx' \; .$$

The CDF for the standard normal is written $N(z)$. It is given by

$$N(z) \;=\; \Pr(Z \le z) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-z'^2/2} \, dz' \; . \tag{1}$$

We use the $N(z)$ notation because there is no explicit formula for the integral. It is related to, but not the same as, the error function.

If $X \sim \mathcal{N}(m, v)$, then

$$\Pr(X \le a) \;=\; \Pr(m + \sqrt{v}Z \le a) \;=\; N\left(\frac{a-m}{\sqrt{v}}\right) \; .$$

Also, $\Pr(Z \leq z) + \Pr(Z \leq z) = 1$. Moreover, the symmetry of the standard normal density implies that $\Pr(Z \geq z) = \Pr(Z \leq -z)$. Together, these imply the relation

$$N(-z) \;=\; 1 \;-\; N(z)\;. \tag{2}$$

If $F(x)$ is the CDF of a random variable then F is an increasing function of x that goes to zero as $x \to -\infty$ and goes to one as $x \to \infty$. For the cumulative normal, there are approximate formulas for these limits

$$N(z) \;\approx\; \frac{1}{\sqrt{2\pi}}\,\frac{1}{|z|}e^{-z^2/2} \quad \text{as} \quad z \to -\infty \tag{3}$$

$$N(z) \;\approx\; 1 - \frac{1}{\sqrt{2\pi}}\,\frac{1}{z}e^{-z^2/2} \quad \text{as} \quad z \to \infty \tag{4}$$

These approximations come from looking at the integral

$$\int_z^\infty e^{-t^2/2}\,dt$$

then making the change of variable $t = z + s$, expanding the exponential as $t^2 = z^2 + 2zs + s^2$, then dropping the s^2:

$$\int_z^\infty e^{-t^2/2}\,dt \;\approx\; e^{-z^2/2}\int_0^\infty e^{-zs}\,ds \;=\; e^{-z^2/2}\,\frac{1}{z}\;.$$

Well, the integrand e^{-zs} should have been $e^{-zs-s^2/2}$. But, and this is what makes the approximation work, if z is large, then the e^{-zs} already approaches zero so quickly that the value of the integral is almost exactly determined before the extra s^2 makes a difference.

2 The continuous time limit

The Black Scholes model is a limit as $\delta t \to 0$ of the binomial tree model. There are several reasons to want δt to go to zero. First, actual trading takes place in nearly continuous time.[1] Second, the continuous time limit would be a good simple approximation even if δt were merely small.

The continuous time limit is a "good" approximation not only in the sense that is reasonably accurate, but also in that it is simpler. The same thing happens in ordinary calculus. The $\delta t \to 0$ limit of a finite difference is the derivative. It may take some time to make this limit rigorous in the mathematical sense,[2] but the formulas and rules of differentiation are simpler than the rules of finite

[1] Trading time on exchanges is measured in milliseconds. On that scale, trading is done in discrete orders. There can be many trades per millisecond, but on this scale it does not look very much like Brownian motion, or the binomial tree model.

[2] Historically, it took about a century. Newton and Leibnitz started working with derivatives and integrals in the late 1600's. Boltzano and Cauchy gave the mathematical definition of a limit in the late 1700's.

differences. The same is true for the $\delta t \to 0$ limit of Riemann sums, which are integrals.

The continuous time limit of the binomial tree process is *geometric Brownian motion*. We will approach this in two steps. This week we discuss only the $\delta t \to 0$ limit of S_T to see that this is a lognormal random variable. That is enough to derive the Black Scholes formula. The stochastic dynamics, Brownian motion, diffusion processes, and Ito calculus, all that is coming in a few weeks.

Now look back to (18) from Week 3 and the discussion there. The log variable at time t_k may be written

$$X_{t_k}^{\delta t} \;=\; X_0 \,+\, \sum_{j=0}^{k-1} Z_j^{\delta t} \;. \tag{5}$$

I have changed notation to explicitly how things depend on δt. X_0 is known and does not depend on δt, but the distribution of the steps $Z_j^{\delta t}$ does. The continuous time limit is the limit $\delta t \to 0$ and $k \to \infty$ with t_k fixed.

The continuous time limit is a limit in *distribution*, which is written

$$X_T^{\delta t} \;\overset{\mathcal{D}}{\to}\; X_T \quad \text{as} \quad \delta t \to 0. \tag{6}$$

This notation does not imply that the actual numbers $X_T^{\delta t}$ converge as $\delta t \to 0$, only the distributions of the random variables. This is related to the $\sim$ notation used above, which might have been written $\overset{\mathcal{D}}{=}$. It means that

$$E\left[V(X_T^{\delta t})\right] \;\to\; E\left[V(X_T)\right] \quad \text{as} \quad \delta t \to 0, \tag{7}$$

at least for reasonable[3] functions V. The distributions of the $X_{t_k}^{\delta t}$, and therefore the distribution of X is in the risk neutral measure of the risk neutral binomial tree.

If the $X_{t_k}^{\delta t}$ have a limiting distribution, the central limit theorem says that it should be Gaussian. This Gaussian has a mean and variance that grow linearly with t:

$$X_{t_k}^{\delta t} \;\overset{\mathcal{D}}{\approx} \mathcal{N}(X_0 + m\,k\delta t, \sigma^2\,k\delta t) \;=\; \mathcal{N}(X_0 + mt_k, \sigma^2 t_k)\;. \tag{8}$$

The mean and variance both are proportional to the number of terms in the sum (5), which is k. If these are to have limits as $\delta t \to 0$, $E[Z_k^{\delta t}]$ and $\text{var}[Z_k^{\delta t}]$ must be proportional to δt. The formula (8) assumes this. The conclusion is that

$$X_T^{\delta t} \;\overset{\mathcal{D}}{\to} X_T \;=\; \mathcal{N}(X_0 + mT, \sigma^2 T) \quad \text{as} \quad \delta t \to 0. \tag{9}$$

It is helpful in calculations to express X_T in terms of a standard normal

$$X_t \;\sim\; X_0 \,+\, mT \,+\, \sigma\sqrt{T}Z \quad \text{where} \quad Z \;\sim\; \mathcal{N}(0,1)\;. \tag{10}$$

[3] The most common notion of "reasonable" in this context is that V should be continuous and bounded. The payout functions we use are not bounded, but grow slowly enough as $|x| \to \infty$ that the limits (7) hold for them too.

Since X_T is the continuous time limit of the log process, $S_T = e^{X_T}$ is the continuous time limit of the lognormal process itself, in the risk neutral measure. Setting $T-0$ gives $S_0 = e^{X_0}$. Therefore, we have

$$S_T \;\sim\; S_0 e^{mT+\sigma\sqrt{T}Z} \;. \tag{11}$$

Now consider a European option that pays $V(S_T)$ at time T. Let $f(s,T)$ be the continuous time limit of the risk neutral price of option price today (at time $t=0$) if the asset spot price is $S_0 = s$. This is given by

$$\begin{aligned} f(s,T) &= e^{-rT} E\left[V(S_T)\right] \\ &= e^{-rT} E\left[V\left(e^{X_T}\right)\right] \\ &= e^{-rT} E\left[V\left(se^{mT+\sigma\sqrt{T}Z}\right)\right] \\ f(s,T) &= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} V\left(se^{mT+\sigma\sqrt{T}z}\right) e^{-z^2/2}\,dz \;. \end{aligned} \tag{12}$$

We said last week that σ, the volatility, is determined from market data. The other parameter, m, is determined by theory. Risk neutral pricing theory states that $E[S_T] = e^{rT}S_0$. One way to see this is to apply the pricing formula to the "option" that pays one unit of stock at time T. If there is no counterparty risk or interest rate risk, there is no difference between having the stock given to you at time T and having it today. Therefore, the price of the "option" must be the price of the stock today. The risk neutral option pricing relation gives $S_0 = e^{-rT}E[S_T]$.

On the other hand, we saw in the first homework assignment that

$$E\left[S_T\right] \;=\; E\left[S_0 e^{mT+\sigma\sqrt{T}Z}\right] \;=\; S_0 e^{\left(m+\frac{\sigma^2}{2}\right)T} \;.$$

Together with the above, this gives $rT = \left(m+\frac{\sigma^2}{2}\right)T$, which implies that $m = r - \frac{\sigma^2}{2}$. Finally we reach the desired general expression

$$f(s,T) \;=\; e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} V\left(se^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z}\right) e^{-z^2/2}\,dz \;. \tag{13}$$

We end this section with some comments on the lognormal random variable

$$S_T \;\sim\; S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z} \;.$$

If T is small, the exponent is close to zero and we can expand in a Taylor series. The first non-trivial term in this Taylor series is a Gaussian:

$$S_T \overset{\mathcal{D}}{\approx} S_0 \,+\, \sigma\sqrt{T}Z \,+\, \left(r-\frac{\sigma^2}{2}\right)T \;.$$

This means that for short time, stocks look roughly Gaussian, in the lognormal model. Looking more carefully at the two terms, when T is small, $\sqrt{T}$ is much larger than T. Therefore, we expect the noise term, $\sigma\sqrt{T}Z$,to *dominate* (be larger than) $\left(r-\frac{\sigma^2}{2}\right)T$, which is the *drift* term. All this is reversed when T is large and the drift terms dominates. In the market today, r is so small that we expect the drift term to be negative. This means that the typical value of S_T is roughly e^{mT}, where m is negative. This says that the typical S_T is exponentially small. You should wonder how a random variable that typically is exponentially small manages to have an exponentially large expected value $E[S_T]=S_0e^{rT}$. The answer is that rare but very large S_T values suffice to bring up the mean. This is what people mean when they talk about the "fat upside tail" of the distribution of S_T for large T.

3 The Black Scholes formula

Black and Scholes gave formulas for the integrals (12) for the cases of vanilla European puts and calls. The prices are called $f(s,T)=P(s,T)$ and $f(s,T)=C(s,T)$ respectively. The put price comes from inserting the put payout $(K-S_T)_+$ into the integral (12). I write $S_T(z)$ if I want to indicate how S_T depends on z. Since the payout is zero for $S_T>K$, the integral over z only needs to go up to Z_*, where $S_T(Z_*)=K$. This equation is

$$K \;=\; se^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}Z_*}\;.$$

Solving gives

$$Z_* \;=\; \frac{\log(K/s)-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\;. \tag{14}$$

With this, we can write the put price as

$$\begin{aligned} P(s,T) \;&=\; e^{-rT}E\left[\left(K\;-\;se^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}Z}\right)_+\right] \\ &=\; e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{Z_*}\left(K\;-\;se^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}z}\right)e^{-z^2/2}\,dz\;, \end{aligned} \tag{15}$$

where Z_* given by (14). The last integral is the sum of two terms, which will become the two terms in the Black Scholes formula. The first one is just

$$Ke^{-rT}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{Z_*}e^{-z^2/2}\,dz \;=\; Ke^{-rT}N(Z_*)\;. \tag{16}$$

The second term is

$$-\,se^{-\frac{-\sigma^2 T}{2}}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{Z_*}e^{\sigma\sqrt{T}z-z^2/2}\,dz\;.$$

We evaluate this integral by completing the square as you did in homework 1:

$$\begin{aligned}\sigma\sqrt{T}z \,-\, z^2/2 &= -\sigma^2 T/2 \,+\, \sigma\sqrt{T}z \,-\, z^2/2 \,+\, \sigma^2 T/2 \\ &= -\left(z-\sigma\sqrt{T}\right)^2/2 \,+\, \sigma^2 T/2\,.\end{aligned}$$

Now the second integral is

$$-s\sqrt{2\pi}\int_{-\infty}^{Z_*} e^{\left(z-\sigma\sqrt{T}\right)^2/2}\,dz\,.$$

The last step is to simplify the exponent in the integrand for the purpose of expressing the answer in terms of the cumulative normal. Therefore, take $w = z - \sigma\sqrt{T}$, which is the same as $z = w + \sigma\sqrt{T}$. To express the upper limit of integration in terms of w, define W_* so that $z = Z_*$ when $w = W_*$. The formula is

$$W_* \;=\; Z_* - \sigma\sqrt{T} \;=\; \frac{\log(K/s) - \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\,. \tag{17}$$

The final form of the second integral is

$$-s\sqrt{2\pi}\int_{-\infty}^{W_*} e^{-w^2/2}\,dw \;=\; -sN(W_*)\,, \tag{18}$$

where W_* is given by (17). Combining the two terms (16) and (18) gives a formula for the put price

$$P(s,T) \;=\; Ke^{-rT}N(Z_*) \,-\, sN(W_*)\,. \tag{19}$$

Now look in Hull (page 291). The formula there is a little different from (19), with (14) and (17). Most people use the formulas in the notation of Hull, so I write them here. It will be a homework exercise to use put/call parity and (2) to see the equivalence.

$$d_1 \;=\; \frac{\log(s/K) \,+\, \left(r \,+\, \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \tag{20}$$

$$d_2 \;=\; \frac{\log(s/K) \,+\, \left(r \,-\, \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \;=\; d_1 \,-\, \sigma\sqrt{T} \tag{21}$$

$$P(s,T) \;=\; Ke^{-rT}N(-d_2) \,-\, sN(-d_1) \tag{22}$$

$$C(s,T) \;=\; sN(d_1) \,-\, Ke^{-rT}N(d_2) \tag{23}$$

It is worthwhile taking the time to understand and interpret these formulas. To start, the N function is dimensionless (being a number between 0 and 1). To get a price (an amount of money), you need to multiply N by a price. For both the call and the put, one term has s and the other has Ke^{-rT}. Both of these represent prices today. K by itself is a price paid in the future. You bring it back to today, to compare it to s, by multiplying by the discount factor e^{-rT}. Note also the units in d_1 and d_2. The argument of the log, s/K, is dimensionless as any proper argument to log should be. Because the log term is dimensionless, the second part of the numerator also must be dimensionless, as rT and $\sigma^2 T$ are. The denominator, $\sigma\sqrt{T}$ also is dimensionless, as we saw last week when talking about the units of vol.

Similar comments concern (20), which may be rewritten as

$$d_1 \ = \ \frac{\log(se^{rT}/K) \ + \ \sigma^2 T/2}{\sigma\sqrt{T}} \ = \ \frac{\log(se^{rT}/K)}{\sigma\sqrt{T}} \ + \ \sigma\sqrt{T}/2 \ . \tag{24}$$

The argument of the log is the ratio of the forward price to the strike price, which is called *moneyness*. The moneyness terms are the same for d_1 and d_2, only the noise correction $1/(\sigma\sqrt{T})$ change signs.

Next, consider how the put and call prices depend on s (more properly, the moneyness). When $s \to 0$, both d_1 and d_2 go to $-\infty$. This makes the arguments in the N functions for P go to ∞ and those in C go to $-\infty$. In this limit, the put price simplifies to $P(s,T) \ \approx \ Ke^{-rT} - s$, which is the the present price of a forward contract with K as the settlement price. That means that when it is far in the money, the put has approximately the same price as the corresponding forward contract. This is natural; it is unlikely that the holder would not wish to exercise the option, so her or his right not to do so is of little value. The limit $s \to 0$ takes the call out of the money, making the call worth very little. $d_{1,2} \to -\infty$ takes both N functions in the call formula (23) to zero.

I end with a final trick for remembering the formulas. An at the money forward contract has strike equal to the forward price, $s = e^{-rT}K$. Both the call and put need to have positive value in that case. The call in that case is $C \ = \ s\left(N(d_1) - N(d_2)\right)$. This happens only if $d_2 < d_1$, since N is an increasing function of d. That is why you subtract something from d_1 to get d_2.

4 Implied vol

Suppose S_0 is the spot price of a stock, and C_M is the market price of a particular call option on this stock with a specific strike and expiration. The *implied vol* is the value of σ that makes (23) equal to C_M. As we discussed last week, there are several reasons to want the Black Scholes implied vol even if the price is already known.

A math person would start by asking: "Do such values of σ exist? Are they unique". It helps to know the derivative with respect to σ. It happens that the σ derivatives of P and C are the same. They are called *Vega* which usually is written Λ (Hull uses a better, but non-standard $\mathcal{V}$.). That is the Greek capital Lambda, which looks like an upside down V. The formula is

$$\frac{\partial P}{\partial \sigma} \ = \ \frac{\partial C}{\partial \sigma} \ = \ \Lambda \ = \ s\sqrt{T}N'(d_1) \ , \tag{25}$$

where (see (1)) $N'(z) = \frac{dN(z)}{dz} = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$. It takes some time to verify this, but it is only algebra.

This formula answers the math questions. Since Vega is positive, there is at most one implied vol for any price. Also, the pricing formulas (22) and (23) show that, for fixed parameter values (s, r, K, and T), both P and C are bounded. For example, $C < s$ always, because $N(d_1) < 1$ and $N(d_2) > 0$. The bottom line is that the implied vol may exist, and if it does exist, it is unique. In practice, market prices almost always lie in the range where the implied vol is defined. In practice, the implied vol is found by solving for σ in the Black Scholes formula using some numerical method.

If the Black Scholes model were correct, the implied vol would be the same for every option on the same underlier. But this is far from true in the marketplace. Instead, the implied vol depends on the strike and expiration in interesting ways. There are many more sophisticated Black Scholes like theories – jump diffusions, stochastic volatility, volatility surfaces, etc. – that you will learn about in future classes.

To begin with, consider options expiring at a fixed time T on the same underlier but with a range of strikes. We consider mostly short dated (small T) options because longer dated options are less liquid. Out of the money options are more interesting than in the money options because in the money options are pretty much the same as forwards, as we saw today and in previous homeworks. Puts are out of the money when the strike is below spot (ignoring e^{rT} factors because T is small or r is small), while calls are out of the money when the strike is above the spot. If you compute the implied vol as a function of the strike, you typically will use puts below the spot and calls above. This implied vol is a strike corrected measure of how expensive the option is. Out of the money options naturally are cheaper than at the money options, but how much cheaper should they be. If an out of the money has a much higher implied vol, you can think of it as being relatively expensive, compared to the at the money option.

Implied vol curves are characterized by *skew* and *smile*. Skew is the overall slope. Equity implied vols usually skew up as the strike decreases. This means that out of the money puts are more expensive (as measured in implied vol) than at the money options or out of the money calls. There are several informal explanations. One invokes the law of supply and demand. The market has a net positive position in any equity, market cap is positive. This means that more people own the stock and are worried about price drops than are short worried about price increases. Out of the money puts are insurance against price drops, so there is more demand for them. Support for this theory comes from currency markets, which have less skew.

Smile is convexity of the implied vol curve. Out of the money options in both directions have higher implied vol than near the money options. In currency markets, you can imagine there are traders on both sides of the currency seeking protection against large moves. Another factor in the smile is that the lognormal model greatly under-estimates the tails of the future price distribution. The asymptotic formula (3) implies that put prices should decay more than exponentially as the strike moves away from the spot. Actual deep out of the money options are more valuable because the actual probabilities of going there are much larger than the lognormal model estimates.

Implied vols often are larger for longer dated options (larger T). One explanation of this (though certainly not the only explanation) is a *stochastic vol* model. Stochastic vol means that the vol itself is changing over time in an unpredictable way. If the vol goes up, then the tails become larger the options become more valuable. The longer the time horizon, the greater the chance the vol has increased so the larger the implied vol for the total time period.

Liquidity is another factor that makes implied vol curves hard to interpret. In most markets there is much less option trading than trading of the underlying asset. The bid/ask spread[4] is larger for options than for the underlier, and the volume is much smaller. The quoted price may be the midpoint between the bid and ask.

[4]Most markets have an order book listing offers, called *limit orders*, to buy or sell at certain prices. The highest price of an offer to buy is the *bid* price. If you want to sell, you either accept the bid price (a *market order*), or you enter a limit order in the order book and hope that a buyer comes along who is willing to pay your price. The lowest offer to sell is the *ask*. The difference between them is the *bid/ask spread*.