Derivative Securities, Fall 2010
Mathematics in Finance Program
Courant Institute of Mathematical Sciences, NYU
Jonathan Goodman
http://www.math.nyu.edu/faculty/goodman/teaching/DerivSec10/resources.html

Week 6

1 The Ito integral

The Black Scholes reasoning asks us to apply calculus, stochastic calculus, to expressions involving differentials of Brownian motion and other diffusion processes. To do this, we define the *Ito integral* with respect to Brownian motion and a general diffusion. Things discussed briefly and vaguely here are discussed in more detail and at greater length in Stochastic calculus.

Suppose X_t is a diffusion process that satisfies (18) and (19) from week 5. Suppose that f_t is another stochastic process that is adapted to the same filtration $\mathcal{F}$. The Ito integral is

$$Y_t \;=\; \int_0^t f_s \, dX_s \; . \tag{1}$$

You could call this the "indefinite integral" because we care how the answer depends on t. In particular, the Ito integral is one of the ways to construct a new stochastic process, Y_t, from old ones f_t and X_t.

It is not possible to define (1) unless f_t is adapted. If f_t is allowed to depend on future values $X_{t'}$ $(t' > t)$, then the integral may not make sense or it may not have the properties we expect.

The essential technical idea in the definition of (1) is that dX_s is in the future of s. If we think that $dX_s = X_{s+ds} - X_s$, then we must take $ds > 0$ (but infinitely close to zero). This implies that if X_t is a martingale, then Y_t also is a martingale. Indeed, following the argument of the last paragraph of week 5, you look at (recall that f_s has a known value given $\mathcal{F}_s$)

$$E\left[\, f_s dX_s \mid \mathcal{F}_s \right] \;=\; f_s \, E\left[\, dX_s \mid \mathcal{F}_s \right] \;=\; 0 \; .$$

The martingale condition is that $E[\, dX_s \mid \mathcal{F}_s] = 0$. At time s, with the information in $\mathcal{F}_s$, the value of f_s is known but the value of the forward looking differential dX_s is not known. The tower property and $\mathcal{F}_t \subseteq \mathcal{F}_s$ implies then that

$$E\left[\, f_s dX_s \mid \mathcal{F}_t \right] \;=\; 0$$

too. Now you get

$$E\left[\, Y_{t'} - Y_t \mid \mathcal{F}_t \right] \;=\; \int_t^{t'} E\left[dX_s \mid \mathcal{F}_t \right] \;=\; 0 \; .$$

The fact that (1) produces a martingale from a martingale is a continuous time version of Doob's theorem . Think of X_t as a tradable asset and f_t as a trading strategy. You cannot make an expected profit trading on a martingale.

There is an official general way to define integrals like (1) that I will not give here.[1] Instead I give a definition of the integral involves the limit as $\delta t \to 0$ of a Riemann sum approximation. (The δt here is not the same as the δt from week 5.) This works when X_t and f_t are diffusions. The approximation is

$$Y_t^{\delta t} = \sum_{0 \le t_j < t} f_{t_j} \left(X_{t_{j+1}} - X_{t_j} \right) . \tag{2}$$

Here, we used the familiar notation $t_j = j\,\delta t$. The theorem is that the limit as $\delta t \to 0$ of the approximations (2) exists in some sense and the limiting process Y_t satisfies $E[dY_t \mid \mathcal{F})t] = 0$ and $\text{var}(dY_t \mid \mathcal{F}_t] = f(t)^2 b(X_t)^2 dt$.

Here is a hint at how it might go. We want to see what happens as $\delta t \to 0$, so we compare the δt and the $\delta t/2$ approximations. The smaller time step $\delta t/2$ cuts each interval (t_j, t_{j+1}) in half. I adopt the notation $t_{j+1/2} = (j + \frac{1}{2})\delta t$. Let $R_t^{\delta t} = Y_t^{\delta t/2} - Y_t^{\delta t}$ be the difference. A calculation shows that

$$R_t^{\delta t} \approx \sum_{t_j < t} \left(X_{t_{j+1}} - X_{t_{j+1/2}} \right) \left(f_{t_{j+1/2}} - f_{t_j} \right) .$$

Let U_j be the general term on the right:

$$U_j = \left(X_{t_{j+1}} - X_{t_{j+1/2}} \right) \left(f_{t_{j+1/2}} - f_{t_j} \right) .$$

If the $R_t^{\delta t}$ go to zero fast enough as $\delta t \to 0$, then it is likely that the limit of the $Y_t^{\delta t}$ exists. For that reason, we calculate $E\left[\left(R_t^{\delta t} \right)^2 \right]$. This is

$$E\left[\left(R_t^{\delta t} \right)^2 \right] = \sum_{t_i < t} \sum_{t_j < t} E\left[U_i U_j \right] = \sum_{j=i} E\left[U_i U_j \right] + \sum_{j \ne i} E\left[U_i U_j \right] .$$

Looking at this shows the martingale and diffusion stuff at work. The second sum is zero and the first sum is small. For the second sum, suppose (without loss of generality) that $t_i > t_j$. Then note that almost everything is known at time $t_{i+1/2}$, so

$$\begin{aligned} E\left[U_i U_j \mid \mathcal{F}_{t_{i+1/2}} \right] &= E\left[\left(X_{t_{i+1}} - X_{t_{i+1/2}} \right) \mid \mathcal{F}_{t_{i+1/2}} \right] \left(f_{t_{i+1/2}} - f_{t_i} \right) \left(X_{t_{j+1}} - X_{t_{j+1/2}} \right) \left(f_{t_{j+1/2}} - f_{t_j} \right) \\ &= 0 . \end{aligned}$$

For the $i = j$ terms, we have

$$\begin{aligned} E\left[U_i^2 \mid \mathcal{F}_{t_{i+1/2}} \right] &= E\left[\left(X_{t_{i+1}} - X_{t_{i+1/2}} \right)^2 \left(f_{t_{i+1/2}} - f_{t_i} \right)^2 \mid \mathcal{F}_{t_{i+1/2}} \right] && (3) \\ &= E\left[\left(X_{t_{i+1}} - X_{t_{i+1/2}} \right)^2 \mid \mathcal{F}_{t_{i+1/2}} \right] \left(f_{t_{i+1/2}} - f_{t_i} \right)^2 && (4) \\ &\approx b^2(X_{t_{i+1/2}}) \delta t \left(f_{t_{i+1/2}} - f_{t_i} \right)^2 && (5) \end{aligned}$$

[1] A popular reference is *Stochastic Differential Equations*, by Bernt Oksendal. A quicker and more precise treatment is *Introduction to Stochastic Integration* by Kai Lai Chung and Ruth Williams.

If f_t is a diffusion, then $E[(f_{t_{i+1/2}} - f_{t_i})^2] \approx \beta^2 \delta t$. Altogether,

$$E\left[R^2\right] \approx \sum_{t_i<t} E\left[b^2\beta^2\right] \delta t^2 = O(t\delta t) . \tag{6}$$

To summarize: If X_t is a martingale, putting the X differences in the future of the integrand, f_{t_i} (i.e. $(X_{t_{i+1}} - X_{t_i}) f_{t_i}$ instead of, for example, $(X_{t_i} - X_{t_{i-1}}) f_{t_i}$) makes it easy to compute expected values because most of them are zero. This is why the off diagonal terms $U_i U_j$ have expected value zero. The difference of a diffusion has variance of order δt. This applies both for X_t and f_t.

The estimate (6) implies that R is of the order of $\sqrt{\delta t}$. This can be turned into an argument (not a complete mathematical proof) that the limit exists based on the following lemma. Suppose A_k is a sequence of numbers and $\sum_{k=1}^{\infty} |A_{k+1} - A_k| < \infty$. Then $\lim_{k\to\infty} A_k$ exists. To apply the lemma here, choose a sequence of time steps $\delta t_k = 2^{-k}$ converging to zero exponentially. Then the corresponding $R \sim 2^{-k/2}$, also go to zero exponentially. Since $Y_t^{\delta t_{k+1}} - Y_t^{\delta t_k} = R_k$, the lemma implies that the limit of the $Y_t^{\delta t_k}$ exists.

Putting the dX in the future is not just a technical trick for the proof, it changes the answer. One famous example of this is the integral

$$I_t = \int_0^t W_s \, dW_s . \tag{7}$$

The correct approximation scheme (2) for this example is

$$I_t \approx I_t^{\delta t} = \sum_{t_j<t} W_{t_j} \left(W_{t_{j+1}} - W_{t_j}\right) . \tag{8}$$

To this we apply a simple trick:[2]

$$W_{t_j} = \frac{1}{2}\left[\left(W_{t_{j+1}} + W_{t_j}\right) - \left(W_{t_{j+1}} - W_{t_j}\right)\right] .$$

This allows us to express $I^{\delta t}$ as $\frac{1}{2}\left(J^{\delta t} + K^{\delta t}\right)$, where

$$\begin{aligned}
J^{\delta t} &= \sum_{t_j<t} \left(W_{t_{j+1}} + W_{t_j}\right)\left(W_{t_{j+1}} - W_{t_j}\right) \\
&= \sum_{t_j<t} \left(W_{t_{j+1}}^2 - W_{t_j}^2\right) \\
&\approx W_t^2 - W_0^2 = W_t^2 \\
K^{\delta t} &= \sum_{t_j<t} \left(W_{t_{j+1}} - W_{t_j}\right)\left(W_{t_{j+1}} - W_{t_j}\right) \\
&= \sum_{t_j<t} \left(\delta W_{t_j}\right)^2 \\
&\approx \sum_{t_j<t} \delta t = t .
\end{aligned}$$

[2] It will be clear how someone (might have) thought up this trick once you see the general theory.

Combining these calculations gives the limit of (8) as

$$I_t \;=\; \frac{1}{2}W_t^2 \;-\; \frac{1}{2}t \; . \tag{9}$$

Other seeming plausible approximations to the Ito integral (7) have limits (as $\delta t \to 0$) that are different from the correct answer (9). An example is to approximate $W_t dW_t$ by $W_{t+\delta t}\left(W_{t+\delta t} - W_t\right) = W_{t+\delta t}\delta W_t$. You can figure out that something is wrong right away by taking the expected value:

$$E\left[W_{t+\delta t}\delta W_t\right] \;=\; E\left[\left(W_t + \delta W_t\right)\delta W_t\right] \;=\; E\left[\left(\delta W_t\right)^2\right] \;=\; \delta t \; .$$

The correct approximation (8) has $E[I_t^{\delta t}] = 0$. The present incorrect approximation has

$$E\left[\sum_{t_j<t} W_{t_{j+1}}\,\delta W_j\right] \;=\; \sum_{t_j<t}\delta t \;\approx\; t \; .$$

In fact, repeating the algebra leading to (9) leads to

$$\lim_{\delta t\to 0}\sum_{t_j<t} W_{t_{j+1}}\,\delta W_j \;=\; \frac{1}{2}W_t^2 \;+\; \frac{1}{2}t \; .$$

The difference between this and the right answer (9) is exactly the expected value t.

2 Ito's lemma

Ito's lemma is something like a stochastic version of the following version of the ordinary chain rule. Suppose $x(t)$ and $y(t)$ are two functions and we construct $F(t) = f(x(t), y(t))$. The differential of F comes from the chain rule

$$dF \;=\; \partial_x f(x,y)dx \;+\; \partial_y f(x,y)dy \; . \tag{10}$$

In ordinary calculus this may be written

$$\frac{dF}{dt} \;=\; \partial_x f(x(t),y(t))\,\frac{dx}{dt} \;+\; \partial_y f(x(t),y(t))\,\frac{dy}{dt} \; . \tag{11}$$

These expressions have an intuitive meaning, but you might have heard that they are not "rigorous".

A rigorous version could involve integration. Clearly it is desirable that, however we define dF, the integral of dF should be the change in F:

$$\int_0^T dF \;=\; F(T) \;-\; F(0) \; . \tag{12}$$

The rigorous meaning of (10) could be

$$F(T) \;-\; F(0) \;=\; \int_0^T \partial_x f(x(t),y(t))\,dx \;+\; \int_0^T \partial_y f(x(t),y(t))\,dy \; . \tag{13}$$

A proof might replace the informal expression (10) with the more formal

$$\delta F \;=\; F(t+\delta t) \;-\; F(t) \;=\; \partial_x f(x,y)\delta x \;+\; \partial_y f(x,y)\delta y \;+\; O(\delta t^2)\;. \qquad (14)$$

Then define $\delta F_j = F(t_{j+1}) - F(t_j)$ (with $t_j = j\delta t$, $\delta t = T/n \to 0$ as $n \to \infty$ with T fixed, as usual) and write the obvious

$$F(T) \;-\; F(0) \;=\; \sum_{j=0}^{n-1} \delta F_j\;.$$

The approximation (14) makes this

$$F(T) - F(0) \;=\; \sum_{t_j<T} \partial_x f(x(t_j), y(t_j))\, \delta x_j + \sum_{t_j<T} \partial_y f(x(t_j), y(t_j))\, \delta y_j \;+\; \text{error}\;,$$

where

$$|\text{error}| \;\leq\; \sum_{t_j<T} O(\delta t^2) \leq T\, O(\delta t) \to 0 \;\text{ as }\; \delta t \to 0\;.$$

The sums on the right are Riemann sum approximations and converge to the integrals on the right side of (13).

Ito's lemma uses all this reasoning plus one extra piece of information. Suppose X_t is a diffusion and we want to find an expression for $df(X_t,t)$. We already saw that in the time interval δt, the increment of X is of order $\sqrt{\delta t}$ (because $E\left[\delta X_t^2\right]$ is of order δt). Therefore, a Taylor series expansion of $\delta f(X_t,t)$ has to include more terms before the error is smaller than order δt. A suitable Taylor expansion is

$$f(X_t{+}\delta X_t, t{+}\delta t) - f(X_t,t) \;=\; \partial_x f(X_t,t)\delta X_t + \frac{1}{2}\partial_x^2 f(X_t,t)\delta X^2 + \partial_t f(X_t,t)\delta t + \text{error}\;.$$

The error terms include $\partial_x^3 f\, \delta X^3$ and $\partial_x^2\partial_t\, f\, \delta X^2 \delta t$, which are order $\delta t^{3/2}$ and δt^2 respectively (in particular, smaller than order δt). Therefore, we have

$$\begin{aligned} f(X_T,T) \;-\; f(X_0,0) \quad &\approx \quad \sum_{t_j<T} \partial_x f(X_{t_j},t_j)\, \delta X_{t_j} \\ &\quad + \quad \frac{1}{2}\sum_{t_j<T} \partial_x^2 f(X_{t_j},t_j)\, \delta X_{t_j}^2 \\ &\quad + \quad \sum_{t_j<T} \partial_t f(X_{t_j},t_j)\, \delta t\;. \end{aligned}$$

The first sum on the right converges to the Ito integral $\int_0^T \partial_x f(X_t,t_j)\, dX_t$. The last term converges to the Riemann integral (the kind from ordinary calculus) $\int_0^T \partial_t f(X_t,t_j)\, dt$.

The middle term is new to Ito. In its limit, you are allowed to replace $\delta X_{t_j}^2$ with its expected value in $\mathcal{F}_{t_j}$. Here is why. Equation (19) from week 5 gives this as $E[\delta X_{t_j}^2 \mid \mathcal{F}_t] = b(X_{t_j})^2 \delta t + \text{error}$. Write the difference as $M_j = \delta X_{t_j}^2 - b(X_{t_j})^2 \delta t$. Then $E[M_j \mid \mathcal{F}_t] \approx 0$ (actually the expected value is the negligable $O(\delta t^2)$. This gives (writing g_t for $\partial_x^2 f(X_{t_j}, t_j)$ for simplicity)

$$\sum_{t_j < T} g_{t_j} \, \delta X_{t_j}^2 \; = \; \sum_{t_j < T} g_{t_j} \, b_{t_j}^2 \delta t \, + \, \sum_{t_j < T} g_{t_j} M_{t_j} \; .$$

The first term converges to the Riemann integral

$$\int_0^T g_t b_t^2 \, dt \; .$$

The second term is small for the same reason $R^{\delta t}$ was small in the previous section. The expected square is

$$E\left[\left(\sum_{t_j < T} g_{t_j} M_{t_j} \right)^2 \right] \; = \; \sum_{ij} E\left[g_{t_i} M_{t_i} \, g_{t_j} M_{t_j} \right]$$

The terms with $i \neq j$ all are (approximately) zero as before This leaves the terms with $i = j$

$$E\left[\left(\sum_{t_j < T} g_{t_j} M_{t_j} \right)^2 \right] \; = \; \sum_{j} E\left[g_{t_j}^2 M_{t_j}^2 \right] \; .$$

But M_j is of order δt, so M_j^2 is the negligibly small order δt^2. Putting this all together gives

$$f(X_T, T) - f(X_0, 0) \; = \; \int_0^T \partial_x f(X_t, t_j) \, dX_t + \frac{1}{2} \int_0^T \partial_x^2 f(X_t, t_j) b^2(X_t) dt + \int_0^T \partial_t f(X_t, t_j) \, dt \tag{15}$$

Written in differential form, this is

$$df(X_t, t) \; = \; \partial_x f(X_t, t_j) \, dX_t \; + \; \frac{1}{2} \partial_x^2 f(X_t, t_j) b^2(X_t) dt \; + \; \partial_t f(X_t, t_j) \, dt \; . \tag{16}$$

It may be easier to remember this if you write $\left(dX_t^2 \right)$ for $E\left[dX_t^2 \mid \mathcal{F}_t \right] = b(X_t)^2 dt$. This standard from of *Ito's lemma* is

$$df(X_t, t) \; = \; \partial_x f(X_t, t_j) \, dX_t \; + \; \frac{1}{2} \partial_x^2 f(X_t, t_j) \left(dX_t^2 \right) \; + \; \partial_t f(X_t, t_j) \, dt \; . \tag{17}$$

We are not saying that $dX_t^2 = b^2 dt$. It isn't. But the expected value of dX_t^2 is $b^2 dt$, which is enough for (15). You might wonder why we cannot replace dX_t with its expected value, which would be zero in the martingale case. The answer is that dX is so much bigger than dX^2 that fluctuations in dX matter while fluctuations in dX^2 do not.

3 Examples

In ordinary calculus the operations are differentiation and integration. The rules of differentiation allow you to calculate the derivative of any algebraic expression. You calculate integrals by trial and error using the rules of differentiation. There are many integrals that cannot be expressed using elementary functions. The cumulative normal $N(z)$ is one of the best known. This is pretty much the situation in stochastic calculus. Ito's lemma allows us to compute differentials. Finding a functional form for an Ito integral boils down to trial and error with Ito's lemma and the basic relation (12).

Consider the example (7). If W_s were an ordinary smooth funcion of s, we could calculate

$$\int_0^t W_s dW_s \;=\; \int_0^t W_s \frac{dW}{ds} ds \;=\; \frac{1}{2}\int_0^t \left(\frac{d}{ds}W_s^2\right) ds \;=\; \frac{1}{2}W_t^2 \;.$$

But Ito's lemma (17) and (13) tell us that $\int_0^t W_s dW_s = \frac{1}{2}W_t^2$ is the same as

$$d\left(\frac{1}{2}W_t^2\right) \;=\; W_t \;,$$

But this is not true. If we take $X_t = W_t$, and $f(x,t) = \frac{1}{2}w^2$, $\partial_w f = w$, $\partial_w^2 f = 1$, and $\partial_t f = 0$, so (17) becomes

$$d\left(\frac{1}{2}W_t^2\right) \;=\; W_t\, dW_t \,+\, \frac{1}{2}\left(dW_t^2\right) \;=\; W_t\, dW_t \,+\, dt \;\neq\; dW_t \;. \tag{18}$$

As we already saw by direct calculation, the Ito integral is not ordinary calculus: $\int_0^t W_s dW_s \neq \frac{1}{2}W_t^2$.

There are several ways to go from the Ito calculation (18) to the right answer. One is to use the correct part of (18) together with (12) to get

$$\frac{1}{2}W_t^2 \;=\; \frac{1}{2}\int_0^t dW_s^2 \;=\; \int_0^t W_s dW_s \,+\, \frac{1}{2}\int_0^t ds \;=\; \int_0^t W_s dW_s \,+\, \frac{1}{2}\,t \;.$$

Rearranging this gives the correct answer (9). Here we used the fact that an integral with respect to ds or dt is an ordinary non-Ito Riemann integral. The definition of the integral in the previous section produces the Riemann integral when the diffusion X_t has the form $a_t dt$ (check this).

This is true even if a_t is random, as long as it is continuous (or Riemann integrable). For example, consider the Ito integral

$$I_t \;=\; \int_0^t W_s^2\, dW_s \;.$$

Motivated by the above, we start by taking the Ito differential of the presumably incorrect calculus answer $f(w,t) = \frac{1}{3}w^3$. The derivatives are $\partial_w f = w^2$ and $\partial_w^2 f = 2w$. Therefore

$$d\left(\frac{1}{3}W_t^3\right) \;=\; W_t^2\, dW \;+\; W_t\, dt \;.$$

This may be rearranged to

$$W_t^2\,dW \;=\; d\left(\frac{1}{3}W_t^3\right) \;-\; W_t\,dt\;,$$

so

$$\int_0^t W_s^2\,dW_s \;=\; \frac{1}{3}W_t^3 \;-\; \int_0^t W_s\,ds\;.$$

The integral term on the right is an ordinary calculus Riemann integral with a random integrand.

Last week we discussed the geometric Brownian motion, which was the solution of the stochastic differential equation

$$dS_t \;=\; \mu S_t dt \,+\, \sigma S_t dW_t\;. \tag{19}$$

As in the above examples, you need to use the Ito calculus to find the correct Ito solution to this. The formula from last week is

$$S_t \;=\; S_0 e^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W_t}\;. \tag{20}$$

We verified that this formula satisfies $E[dS_t \mid \mathcal{F}_t] = \mu S_t dt$ and $E[(dS_t)^2 \mid \mathcal{F}_t] = \sigma^2 S_t^2 dt$. Now we can use Ito's lemma (17) to verify that the formula (20) satisfies the SDE (19). The function is $f(w,t) = S_0 e^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma w}$ with derivatives $\partial_w f = \sigma f$, $\partial_w^2 f = \sigma^2 f$, and $\partial_t f = \left(\mu - \frac{\sigma^2}{2}\right) f$. Therefore (writing the various terms on separate lines)

$$\begin{aligned} d\left(S_0 e^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma w}\right) \;&=\; \sigma S_0 e^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma w}\,dW_t \\ &\quad+\; \frac{1}{2}\sigma^2 S_0 e^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma w}\,dt \\ &\quad+\; \left(\mu-\frac{\sigma^2}{2}\right) S_0 e^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma w}\,dt\;. \end{aligned}$$

You can check that this is the same as (19).

Ito's lemma applied to diffusions other than Brownian motion. The Black Scholes argument applies it to the goemetric Brownian motion. To prepare for that consider the simple example $\int_0^T S_t dS_t$. (I changed notation to avoid writing $S_s dS_s$.) Now $(dS^2) = \sigma^2 S^2 dt$, so the Ito calculation is

$$dS_t^2 \;=\; 2S_t dS_t \,+\, \sigma^2 S_t^2 dt\;.$$

As above, this leads to

$$\int_0^T S_t\,dS_t \;=\; \frac{1}{2}S_T^2 \,-\, \sigma^2 \int_0^T S_t^2\,dt\;.$$

4 The Black Scholes argument

The original pricing argument of Black and Scholes did not use the binomial tree. Instead it used the following form of the arbitrage/replication argument. Let Π_t be the value of an actively managed portfolio but self financing portfolio. Suppose Π_t is a diffusion process with zero noise, which means that $E[d\Pi \mid \mathcal{F}_t] = a_t dt$ and $E[(d\Pi)^2 \mid \mathcal{F}_t] = 0$. Then Π is risk free, which implies that it grows at the risk free rate: $d\Pi_t = r\Pi_t dt$.

This argument assumes that trading takes place in continuous time and that you can have any amount of stock, provided you pay the market price. It assumes that the stock price, S_t, is a geometric Brownian motion (19). A technically correct version of the argument is a little involved[3] The argument presented by Black and Scholes, which I present below, was slightly incomplete or incorrect, depending on how gentle a grader you are. The issue is how you model the self financing aspect. We did this carefully in the binomial tree model, but not here.

The *Delta hedge* is a portfolio consisting of one option and a short position of Δ units of stock. The value of the option at time t is $f(S_t, t)$. Part of the Black Scholes argument is that there is such a pricing function $f(s,t)$ so that the option price is completely determined by the stock price and the time to expiration. I return to this point below. But for now, please accept that there is such a pricing function. The time variable, t, is calendar time, not the time to expiry, which is $T - t$. The value of the portfolio at time t is

$$\Pi_t \;=\; f(S_t, t) \;-\; \Delta_t S_t \; . \tag{21}$$

The informal argument of Black and Scholes asks us to calculate $d\Pi$ holding Δ_t fixed. The argument is that you buy a hedge and hold it for time dt while the market moves. Of course, Δ must be determined by the information in $\mathcal{F}_t$. Hedging knowing the future would have higher returns. Applying Ito's lemma (17) gives

$$d\Pi_t \;=\; \partial_s f(S_t,t)\, dS_t \;+\; \frac{1}{2}\partial_s^2 f(S_t,t)(dS_t)^2 \;+\; \partial_t f(S_t,t)\, dt \;-\; \Delta_t \, dS_t \; .$$

The noise term is eliminated by the choice

$$\Delta_t \;=\; \partial_s f(S_t, t) \; . \tag{22}$$

From (19) we get $(dS_t)^2 = \sigma^2 S_t^2 dt$, so

$$d\Pi_t \;=\; \left(\frac{\sigma^2 S_t^2}{2} \partial_s^2 f(S_t,t) \;+\; \partial_t f(S_t,t) \right) dt \; .$$

Since this has zero noise, Black and Scholes argue that it is equal to $r\Pi_t dt$. Putting in (22) into (21), that is

$$r\Big(f(S_t,t) \;-\; \partial_s f(S_t,t) S_t \Big)\, dt \;=\; \left(\frac{\sigma^2 S_t^2}{2} \partial_s^2 f(S_t,t) \;+\; \partial_t f(S_t,t) \right) dt \; .$$

[3]See, for example, *Martingale Methods in Financial Modeling* by Marek Musiela and Marek Rutkowski.

Combining terms leads to the *Black Scholes equation*

$$0 \;=\; \partial_t f \;+\; \frac{\sigma^2 S_t^2}{2}\,\partial_s^2 f(S_t,t) \;+\; rs\partial_s f \;-\; rf \; . \tag{23}$$

The PDE (*partial differential equation*) (23) determines option prices in much the same way the binomial tree does. You specify the value of the option at time T, the expiration time, then use (23) to "march" backward in time toward the present. We will talk about that process in more detail next week.

There are two simple solutions to (23) that you can use to check that you have the equation right. The first is the option that pays one dollar at time T no matter what. The value at time $t < T$ of this is $f(s,t) = e^{-r(T-t)}$. This satisfies (23) because $\partial_t f = rf$ (note the plus sign) and all the s derivatives are zero. The second is the option that pays one share of stock at time T. Having one share of stock at time T is the same as having one share at time $t < T$. Therefore $f(s,t) = s$ should be a solution, and it is.