Derivative Securities -- Spring 2007 — Section 1

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Forwards, puts, calls, and other contingent claims. This section discusses the most basic examples of contingent claims, and explains how considerations of arbitrage determine or restrict their prices. This material is in chapters 1 and 3, sections 8.1 and 8.2, and chapter 10 of Hull (6th edition). We concentrate for simplicity on European options rather than American ones, on forwards rather than futures, and on deterministic rather than stochastic interest rates. The pricing of forwards is then developed in some detail, corresponding to chapter 5 in Hull.

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The most basic instruments:

Forward contract with maturity T and delivery price K.

buy a forward ↔ hold a long forward
↔ holder is obliged to buy the underlying asset at price K on date T.

European call option with maturity T and strike price K.

buy a call ↔ hold a long call
↔ holder is entitled to buy the underlying asset at price K on date T.

European put option with maturity T and strike price K.

buy a put ↔ hold a long put
↔ holder is entitled to sell the underlying asset at price K on date T.

These are *contingent claims*, i.e. their value at maturity is not known in advance. Payoff formulas and diagrams (value at maturity, as a function of S_T = value of the underlying) are shown in the Figure.

Any long position has a corresponding (opposite) *short* position:
Buyer of a claim has a long position ↔ seller has a short position.

Payoff diagram of short position = negative of payoff diagram of long position.

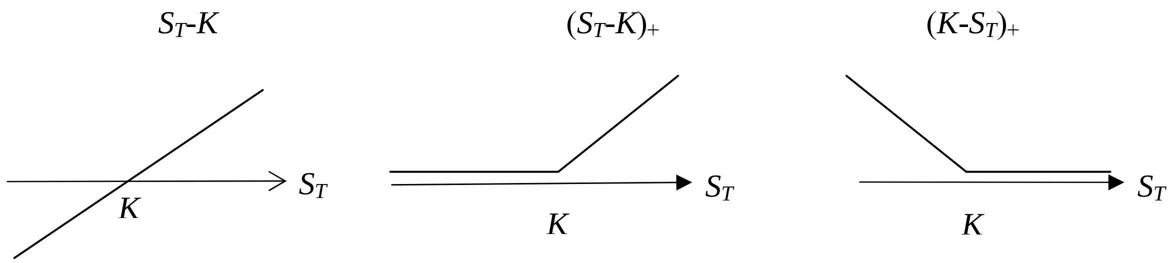

Figure 1: Payoffs of forward, call, and put options

An *American* option differs from its European sibling by allowing early exercise. For example: the holder of an American call with strike K and maturity T has the right to purchase the underlying for price K at any time $0 \le t \le T$. A discussion of American options must deal with two more-or-less independent issues: the unknown future value of the underlying, and the optimal choice of the exercise time. By focusing initially on European options we'll develop an understanding of the first issue before addressing the second.

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Why do people buy and sell contingent claims? Briefly, to *hedge* or to *speculate*.
Examples of hedging:

- A US airline has a contract to buy a French airplane for a price fixed in Euros, payable one year from now. By going long on a forward contract for Euros (payable in dollars) it can eliminate its foreign currency risk.

- The holder of a forward contract has unlimited downside risk. Holding a call limits the downside risk (but buying a call with strike K costs more than buying the forward with delivery price K). Holding one long call and one short call costs less, but gives up some of the upside benefit:

$$(S_T - K_1)_+ - (S_T - K_2)_+ \qquad K_1 < K_2$$

  This is known as a ``bull spread". (See the figure.)

Options are also frequently used as a means for speculation. Basic reason: the option is more sensitive to price changes than the underlying asset itself. Consider for example a European call with strike K=50, at a time t so near maturity that the value of the option is essentially $(S_t - K)_+$. Let $S_t = 60$ now, and consider what happens when S_t increases by 10% to 66. The value of the option increases from about $60 - 50 = 10$ to about $66 - 50 = 16$, an increase of 60%. Similarly if S_t decreases by 10% to 54, the value of the option decreases from 10 to 4, a loss of 60%. This calculation isn't special to a call: almost the same calculation applies to stock bought with borrowed funds. Of course there's a difference: the call has more limited downside exposure.

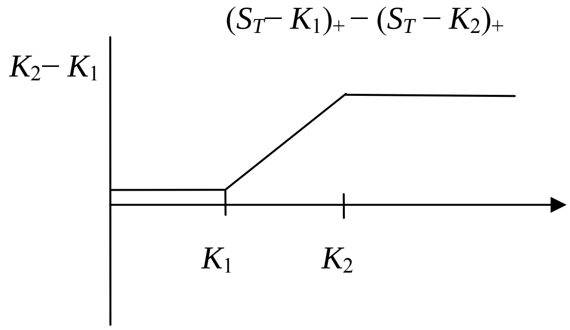

Figure 2: Payoff of a bull spread

We assumed the time t was very near maturity so we could use the payoff $(S_T - K)_+$ as a formula for the value of the option. But the idea of the preceding paragraph applies even to options that mature well in the future. We'll study in this course how the Black-Scholes analysis assigns a value $c = c[S_t; T - t, K]$ to the option, as a function of its strike K, its time-to-maturity $T - t$ and the current stock price S_t. The graph of c as a function of S_t is roughly a smoothed-out version of the payoff $(S_t - K)_+$.

Don't be confused: our assertion that "the option is more sensitive to price changes than the underlying asset itself" does *not* mean that $\partial c/\partial S$ is bigger than 1. This expression, which gives the sensitivity of the option to change in the underlying, is called Δ. At maturity the call has value $(S_T - K)_+$ so $\Delta = 1$ for $S_T > K$ and $\Delta = 0$ for $S_T < K$. Prior to maturity the Black-Scholes theory will tell us that Δ varies smoothly from nearly 0 for $S_t << K$ to nearly 1 for $S_t >> K$.

Forward contracts can also be used for speculation. Holding a portfolio of assets accomplishes two things – (1) it is a place to invest money you don't need now to meet future needs (e.g., saving for retirement) and (2) you invest in assets you think will increase in value. But suppose you only want to accomplish the second objective and don't have need of the first. Forwards give you a means to take positions in assets you think will increase in value without tying this to the investment of cash. Equally important, they give you a means for taking a position in assets you think will go down in value or taking positions that reflect views on relative performance (long one set of assets and short another set; you may not have an opinion as to whether either will increase or decrease in value but you do believe the first set will outperform the second).

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The most critical theorem to be proved in this course – the most important idea in the mathematical theory of derivative securities – is that the current price of a derivative security should be present value of its expected value under a "risk-neutral" expectation. In symbols,

$$V_0 = e^{-rT} E_{RN}[V_T]$$

What does this mean? It means that you take a probability-weighted expected value of final-time payoffs, but that the probabilities used do not represent anyone's beliefs about the future but instead are adjusted so that the expected value of returns have the same expected value as the returns from a "risk-free" asset. And the discounting to present value is done at the discount rate for payments which are certain, the "risk-free" interest rate. So we are ignoring two factors which we assume will exactly offset, we are ignoring the higher expected return that is most likely for a risky asset, and we are ignoring the extra discounting that an investor will expect for taking on this risk.

How can we be certain that those two effects will exactly offset? The type of argument that we offer is one that produces a possible arbitrage strategy — one that can make money for sure. If one of the two effects is stronger than the other, then we demonstrate an arbitrage strategy that will make money from the difference. It is only when the two effects exactly offset that there are no arbitrage opportunities — what we label an "arbitrage-free" state. The theoretical demonstration of how to make potential arbitrage profits is just as important as the determination of an arbitrage-free price. When one of these two effects is stronger than the other, real arbitrage profits can be made. There are individuals and firms whose job is to make such profits and they utilize the theory we expound here to guide their actions. When they are effective, they bring prices back in line with an exact offset in the two effects, and everyone else in the market can go about pricing on the assumption of arbitrage-free conditions.

We will prove this theorem several times, under several different sets of assumptions. Today we will prove it for forward contracts. The required arbitrage strategy for these is quite simple and does not involve elaborate assumptions. Over the next several weeks, we will prove this theorem for options. This requires more complex assumptions and more complex arbitrage strategies. We will wind up proving this theorem for options three different times, under three different sets of assumptions.

Much of the rest of the course will be taken up with specific solution techniques for this equation. For standard call and put options, we'll use this equation to derive the classic Black-Scholes formula, using simple probability calculations. For more complex options, we'll demonstrate Monte Carlo and tree-building techniques to solve for prices. For interest rate options such as caps, floors, and swaptions, we'll derive other results similar to the Black-Scholes formula, from this equation.

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Some pricing principles:

- If two portfolios have the same payoff then their present values must be the same.
- If portfolio 1's payoff is always at least as good as portfolio 2's, then present value of portfolio 1 $\geq$ present value of portfolio 2.

We'll see presently that these principles must hold, because if they didn't the market would support arbitrage.

First example: value of a forward contract. We assume for simplicity:

a. underlying asset pays no dividend (e.g., a non-dividend-paying stock);
b. time value of money is computed using compound interest rate r, i.e. a guaranteed income of D dollars time T in the future is worth $e^{-rT}D$ dollars now.

The latter hypothesis amounts to introducing one more investment option:

Bond worth D dollars at maturity T

buy a bond	↔	hold a long bond
	↔	lend $e^{-rT}D$ dollars, to be repaid at time T with interest.

Consider these two portfolios:

Portfolio 1 − one long forward with maturity T and delivery price K, payoff $(S_T - K)$.

Portfolio 2 − long one unit of stock (present value S_0, value at maturity S_T) and short one bond (present value $-Ke^{-rT}$, value at maturity $-K$).

They have the same payoff, so they must have the same present value. Conclusion:

$$\text{Present value of forward} = S_0 - Ke^{-rT}.$$

In practice, forward contracts are normally written so that their present value is 0. This fixes the delivery price, known as the *forward price*:

$$\text{forward price} = S_0\, e^{rT} \text{ where } S_0 \text{ is the spot price}$$

We can see why the ``pricing principles" enunciated above must hold. If the market price of a forward were different from the value just computed then there would be an arbitrage opportunity:

forward is overpriced	→	sell portfolio 1, buy portfolio 2
	→	instant profit at no risk
forward is underpriced	→	buy portfolio 1, sell portfolio 2
	→	instant profit at no risk.

In either case, market forces (oversupply of sellers or buyers) will lead to price adjustment, restoring the price of a forward to (approximately) its no-arbitrage value.

Two pieces of financial markets terminology that you may find confusing:

(1) Individuals commonly only borrow and lend one type of financial instrument – currency (e.g. dollars). But financial institutions borrow and lend all types of financial instruments. So in the above example, when we talked about "sell portfolio 2," this involves borrowing the stock at time 0 and selling it, then buying the stock at time T in order to repay the borrowing. This is the same mechanism that is used when "short sellers" want to put on a position that will benefit from a decline in stock prices.

(2) Holding a bond is used as a synonym for lending money. Shorting a bond is used as a synonym for borrowing money. So in the above example, when we talk about buying portfolio 2, this means buying the stock and borrowing the money to buy the stock, and borrowing money is equivalent to shorting the bond (to see this consider that borrowing involves receiving Ke^{-rT} dollars now and paying K dollars at time T; if you borrow the bond now you can sell it for Ke^{-rT} dollars and at time T repay the bond and receive K dollars, the bond's principal). When we talk about selling portfolio 2, this means borrowing the stock to sell it and lending the money received from the sale until time T, and lending money is equivalent to buying the bond (to see this, consider that lending involves paying Ke^{-rT} dollars now and receiving K dollars at time T; if you buy the bond now for Ke^{-rT} dollars you can redeem it at time T for the principal amount of K dollars).

Second example: put-call parity. Define

$p[S_0,T,K]$ = price of European put when spot price is S_0, strike price is K, maturity is T

$c[S_0,T,K]$ = price of European call when spot price is S_0, strike price is K, maturity is T.

The Black-Scholes model gives formulas for p and c based on a certain model of how the underlying security behaves. But we can see now that p and c are related, without knowing anything about how the underlying security behaves (except that it pays no dividends and has no carrying cost). ``Put-call parity" is the relation

$$c[S_0,T,K] - p[S_0,T,K] = S_0 - Ke^{-rT}.$$

To see this, compare

Portfolio 1 − one long call and one short put, both with maturity T and strike K; the payoff is $(S_T - K)_+ - (K - S_T)_+ = S_T - K$.

Portfolio 2 − a forward contract with delivery price K and maturity T. Its payoff is also $S_T - K$.

These portfolios have the same payoff, so they must have the same present value. This justifies the formula.

Third example: The prices of European puts and calls satisfy

$$c[S_0,T,K] \geq (S_0 - Ke^{-rT})_+ \quad \text{and} \quad p[S_0,T,K] \geq (Ke^{-rT} - S_0)_+$$

To see the first relation, observe first that $c[S_0,T,K] \geq 0$ by optionality − holding a long call is never worse than holding nothing. Observe next that $c[S_0,T,K] \geq S_0 - Ke^{-rT}$, since holding a long call is never worse than holding the corresponding forward contract. Thus $c[S_0,T,K] \geq \max\{0, S_0 - Ke^{rT}\}$, which is the desired conclusion. The argument for the second relation is similar.

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Note some hypotheses underlying our discussion:

- no transaction costs — no bid-ask spread;
- no tax considerations;
- unlimited possibility of long and short positions — no restriction on borrowing;
- same cost for borrowing money and lending money;
- no charge for borrowing securities.

These are of course merely approximations to the truth (like any mathematical model). More accurate for large institutions than for individuals.

Note also some features of our discussion: We are simply reaping consequences of the hypothesis of no arbitrage. Conclusions reached this way don't depend at all on what you think the market will do in the future. Arbitrage methods restrict the prices of (related) instruments. On the other hand they don't tell an individual investor how best to invest his money. That's the issue of portfolio optimization, which requires an entirely different type of analysis and is discussed in the course Capital Markets and Portfolio Theory.

More realistic assumptions about borrowing stock. We have implicitly been assuming up till now that the stock can be borrowed without paying a borrowing fee (since we assumed that we could currently sell the stock at price S_0 and buy it at time T at S_T and have not talked about any cost for borrowing the stock during that period). How realistic is that assumption? It's not totally unrealistic, because we have been assuming a non-dividend-paying stock, so we don't need to pay the stock lender for missing out on dividends. We don't have to pay the stock lender for any credit risk that we don't pay her back, because we can use the cash we receive for selling the stock as collateral to assure the lender she will receive the stock back. The holder can't expect to be compensated for having her money tied up in the stock – the expected increase in stock price is supposed to be her return for that. But even with these considerations, it is common for holders of stock to require some fee for lending their stock (if for no other reason than that holders of a stock, who want to see the price go up, need some compensation for aiding short sellers who by selling stock are helping to drive the price of the stock down). The fee may be quite small – a rate of ½% per year is not uncommon – but can also be considerably larger, if there is a big demand for borrowing the stock (e.g., many people desiring to sell the stock short finding it is hard to locate stock to borrow).

What is the impact of this fee? It lowers the cost of buying portfolio 2, since as holder of the stock, you can lend it out and earn the stock borrowing fee, and raises the cost of selling portfolio 2, since you need to pay the borrowing rate of q. If we assume an annualized continuously compounded stock borrowing rate of q, the size of the borrowing charge is $S_0(1 - e^{-qT})$, the profit from being long portfolio 2 is now $(S_0 - Ke^{-rT}) - S_0(1 - e^{-qT}) = S_0e^{-qT} - Ke^{-rT}$, the impact is to make the present value of the forward $S_0e^{-qT} - Ke^{-rT}$ and the forward price that makes this present value 0 be $K = S - e^{(r-q)T}$. (Compare with Hull's discussion in section 5.6 on forwards on assets with known yield.)

How do we handle a dividend paying stock? Dividends are paid at fixed dates (usually once a quarter) and are not known in advance. For simplicity, we will approximate this effect by assuming that the market can project dividends with good accuracy (a reasonable assumption over short time periods), while noting that any uncertainty will widen the bounds within which arbitrage can determine the forward price. We will approximate the dividend by an annualized, continuously compounded rate q.

Instead of constructing portfolio 2 by buying one unit of the stock, we will buy e^{-qT} units of the stock and continuously reinvest all dividends in the stock. By the end of period T, this will result in our holding exactly the one unit of the stock which we need to deliver into the forward contract of portfolio 1. If we are selling portfolio 2, we borrow e^{-qT} units of the stock and continuously borrow more units at the dividend rate q. (A fully detailed version of this approach can be found in Baxter & Rennie, p. 107. Also see Hull, section 5.6).

When we have both borrowing costs of the stock and dividend, we just use a q which represents the dividend rate plus the borrowing cost. (Institutional detail — the contract between the stock borrower and lender can either call for the borrower to return to the lender (1) just the stock or (2) the stock plus all dividends paid during the borrowing period. In the first case, the borrowing rate paid will equal the expected dividend rate plus a borrowing add-on, to compensate the lender for missing dividends. In the second case, the borrowing rate should be just the same as in our non-dividend-paying stock case, but the actual payment by the borrower will include the dividend.)

In general, for any asset on which q is the rate of return (annualized and continuously compounded), the present value of a forward at a settlement price of K given on an asset currently priced at S_0 is $S_0e^{-qT} - Ke^{-rT}$ and the settlement price that makes this present value = 0 is $K = S_0e^{(r-q)T}$. Our original example of a non-dividend-paying stock with a zero borrowing cost is just a special case of these formulas, with q =0.

We can now demonstrate our first example of the general formula $V_0 = e^{-rT}\,\mathrm{E}_{\mathrm{RN}}[V_T]$. The "risk-neutral" probability distribution will be defined as the one for which $S_0e^{-qT} = e^{-rT}E_{RN}[S_T]$, so that $E_{RN}[S_T] = S_0e^{(r-q)T}$. $E_{RN}[K] = K$, since this is not a stochastic process. Hence,

$$\begin{aligned} e^{-rT}\,\mathrm{E}_{\mathrm{RN}}[V_T] &= e^{-rT}\,\mathrm{E}_{\mathrm{RN}}[S_T - K] \\ &= e^{-rT}(S_0\, e^{(r-q)T} - K) \\ &= S_0\, e^{-qT} - Ke^{-rT} \end{aligned}$$

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Let's now examine briefly some more specific applications of this general approach.

Forwards on stock indices. The stock index forward can be arbitraged by a portfolio of all the individual stocks in the index with asset weights equal to those of the index (Hull 5.9). Dividends and borrowing costs of this basket can be estimated as weighted averages of the dividends and borrowing costs of each individual stock.

Forwards on foreign exchange rates. Say you have a forward agreement to exchange X units of one currency for KX units of another currency (example X = 1MM, K = 1.25MM, you have a forward agreement to exchange 1MM Euros for 1.25MM Dollars). Buying Portfolio 2 for the arbitrage consists of (1) buying 1MM e^{-qt} Euros now, investing them at the Euro lending rate of q and having 1MM Euros at the end of time T, and (2) borrowing 1.25MM e^{-rt} Dollars now, borrowed at the Dollar lending rate of r, and having 1.25MM Dollars at the end of time T. Since this is equivalent at the end of T to the forward, the present value of the forward must equal the current cost of Portfolio 2, which is 1MM $e^{-qt} S_0$ – 1.25MM $e^{-r\mathrm{T}}$, where S_0 is the current exchange rate for Euros into Dollars. This is clearly just an example of our general formula. Compare with Hull 5.10. The forward exchange rate that makes this present value equal to 0 is $S_0e^{(r-q)T}$.

Forward contract to exchange one asset for another. There's nothing special about one of the assets in this last example being dollars. We could consider a forward contract to exchange Yen for Euros and its present value, by the exact same reasoning as above, would be $e^{-qT}S_0$ - $e^{-rT}K$, where S_0 is the current exchange rate between Yen and Euros, K the forward exchange rate, q the borrowing rate for Euros, and r the borrowing rate for Yen. This formula gives the present value in Yen, so to get a Dollar present value you only need to multiply by the current Dollar/Yen exchange rate. We could equally well value an exchange contract between any two assets: gold and diamonds, oil and cattle, oil and Yen, etc. Compare with Hull 22.11, which is about options to exchange one asset for another, but the analogous reasoning clearly applies to forwards.

Forwards on commodities. Traditional treatments, such as Hull 5.11, distinguish between commodities that are primarily investment assets, such as gold and silver, and commodities that are primarily utilized for consumption, such as oil. A useful comparison would be to non-dividend-paying stocks and currencies. An investment asset, like a non-dividend-paying stock, can be expected to have a very low borrowing rate, q, because the only demand to borrow the asset comes from those wishing to sell it short (it might be even lower for an asset like gold that has associated storage costs which the borrower is taking over from the investor). A consumption asset, like a currency, can be expected to have a relatively high borrowing cost since there is diversified demand to borrow it. Unlike the currency, there is usually no direct way to see the borrowing cost of a commodity – it needs to be backed out of the forward price.

Investment assets, since their borrowing rate, q, is almost always lower than the risk-free interest rate (on currency), r, will almost always have a forward price $S_0e^{(r-q)T}$ that is higher than the current price. For commodities, his situation is known as *contango*. A consumption asset might well have its borrowing cost, q, be higher than the risk-free intetest rate, r, hence its forward price $S_0e^{(r-q)T}$ may be lower than the current price. For commodities, this situation is known as *backwardation*.

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A word about interest rates. In the real world interest rates change unpredictably. And the rate depends on maturity. In discussing forwards and European options this isn't particularly important: all that matters is the cost ``now" of a bond worth one dollar at maturity T. Up to now we wrote this as e^{-rT}. When multiple borrowing times and maturities are being considered, however, it's clearer to use the notation

$$B(t,T) = \text{cost at time } t \text{ of a risk-free bond worth 1 dollar at time } T.$$

In a constant interest rate setting $B(t,T) = e^{-r(T-t)}$. If the interest rate is non-constant but deterministic − i.e. known in advance − then an arbitrage argument shows that $B(t_1,t_2)B(t_2,t_3) = B(t_1, t_3)$. If however interest rates are stochastic − i.e. if $B(t_2,t_3)$ is not known at time t_1 − then this relation must fail, since $B(t_1,t_2)$ and $B(t_1,t_3)$ are (by definition) known at time t_1.

Since our results on forwards, put-call parity, etc. used only one-period borrowing, they remain valid when the interest rate is nonconstant and even stochastic. For example, the value at time 0 of a forward contract with delivery price K is $S_0 - KB(0,T)$ where S_0 is the spot price. We could also use the notation r_T for the interest rate that is applicable to period T and write $B(0,T) = e^{-r_T T}$. For two different time periods T_1 and T_2, we do **not** generally expect that $r_{T_1} = r_{T_2}$.

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Forwards versus futures. A future is a lot like a forward contract − its writer must sell the underlying asset to its holder at a specified maturity date. However there are some important differences:

- Futures are standardized and traded, whereas forwards are not. Thus a futures contract (with specified underlying asset and maturity) has a well-defined ``future price" that is set by the marketplace. At maturity the future price is necessarily the same as the spot price.
- Futures are ``marked to market," whereas in a forward contract no money changes hands till maturity. Thus the value of a future contract, like that of a forward contract, varies with changes in the market value of the underlying. However with a future the holder and writer settle up daily while with a forward the holder and writer don't settle up till maturity.

The essential difference between futures and forwards involves the timing of payments between holder and writer: daily (for futures) versus lump sum at maturity (for forwards). Therefore the difference between forwards and futures has a lot to do with the time value of money. If interest rates are constant − or even nonconstant but deterministic − then an arbitrage-based argument shows that the forward and future prices must be equal. (I like the presentation in the appendix to chapter 5 of Hull. It is presented in the context of a constant interest rate, but the argument can easily be modified to handle a deterministically-changing interest rate.)

If interest rates are stochastic, the arbitrage-based relation between forwards and futures breaks down, and forward prices can be different from future prices. In practice they are different, but usually not much so. Later in the course, when we discuss interest rate derivatives, we'll spend more time on the difference between futures and forwards, including proving the equality of forward and futures prices when interest rates are deterministic and looking at how the relationship changes when interest rates are stochastic. Until we reach that part of the course, we'll ignore the difference and treat forwards and futures as synonymous.