

Derivative Securities -- Spring 2007 — Section 2

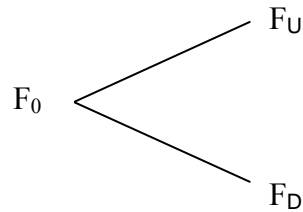
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Binomial single-period options models

Historically, Black, Scholes, and Merton first developed their arbitrage-based approach to pricing options for the simple case of an option for non-dividend paying stock. Over time this approach was then extended to other classes of options – options on dividend paying stocks, stock indices, foreign exchange, exchange between two assets, interest rate products, and futures and forwards. The standard approach to teaching this material is to follow the line of historical development – first develop a model for non-dividend-paying stock, then extend the model to cover other types of assets. I am going to follow a different path. I will first develop a model for options on forwards without specifying what underlying asset the forward references. Then I will derive all the formulas for special cases as easy consequences of this model for options on forwards and the work we have already done on pricing different types of forwards. The reasons I have chosen this treatment are:

- (1) I think it brings out the form of the argument more clearly. You only need to learn a single version of the hedging argument for options, the most difficult part of the subject. Everything else can be explored in the easier setting of hedging forwards.
- (2) It is often the case that the best hedges for options are forwards or futures. When historical data is available for the volatility of forwards or futures, this leads directly to option formulas that incorporate the full volatility of the underlying, including the volatility due to interest rates. Even when historical data for forwards volatility is not available, it can always be created using the formulas we have already developed relating forward prices to cash prices and to interest rates.

In last week's examples of relative pricing, we relied on specific features of the instruments involved. Today, we are going to be looking at a different case, one in which we rely more on the structure of possible price evolution than on specific instrument structure. In particular, we are going to assume that there are only two possible "states of the world" at some future time T . We are trying to find a price for some instrument, V , for which we do not have a direct market price available. We know what the payoffs will be for V at the end of time T — they will be V_U and V_D and we will choose our labeling so that $V_U \geq V_D$. We are looking to determine a price V_0 that should be paid now for the potential payoffs. We will see if we can utilize relative pricing based on a marketable instrument, F , which has a current forward price of F_0 and future payoffs at the end of time T of F_U and F_D .



Prices in the one-period binomial market model.

We could suppose we know the probability p that the forward will be worth F_U at time T . This would allow us to calculate the expected value of any contingent claim. However we will make no use of such knowledge. Pricing by arbitrage considerations makes no use of information about probabilities -- it uses just the list of possible events.

Even before considering the use of pricing relative to F , we can conclude that V_0 must be no greater than $e^{-rt}V_U$ and no less than $e^{-rt}V_D$ by arbitrage considerations. (If it were less than $e^{-rt}V_D$, you could borrow money to purchase the instrument, pay back V_D at the end of time t and receive at least V_D in payoff. If it were greater than $e^{-rt}V_U$, you could sell the instrument, invest the proceeds, receive back more than V_U at the end of time T and pay off at most V_U on the instrument.) Another way of stating this is:

$V_0 = e^{-rt}(V_D + q(V_U - V_D))$ where $0 \leq q \leq 1$; which is algebraically equivalent to

$V_0 = e^{-rt}(q V_U + (1 - q) V_D)$ where $0 \leq q \leq 1$

$qV_U + (1 - q)V_D$, with $0 \leq q \leq 1$, has the mathematical form of a probability mixture of V_U and V_D . This will be important to us in later lectures, allowing us to take advantage of the computational power of probability theory, in particular of the Central Limit Theorem.

Let's get back to trying to price V relative to F . Suppose we initially invest in one unit of V and some amount of F , which we'll call ϕ . The initial cost will be V_0 and the payoffs will be either $V_U + \phi(F_U - F_0)$ or $V_D + \phi(F_D - F_0)$. We can take advantage of there only being two possible final states to set ϕ in a way that makes the payoff amount a certainty -- the same in both states U and D . We'll set $V_U + \phi(F_U - F_0) = V_D + \phi(F_D - F_0)$ and get

$\phi = \frac{V_U - V_D}{F_D - F_U}$. Now, since investing V_0 gets us a known payoff at the end of T , it is

equivalent to a fixed rate investment that returns $V_D + \phi(F_D - F_0)$ ($= V_U + \phi(F_U - F_0)$). By the law of one price, the current price of this investment, V_0 , must be equal to the current price of the fixed rate investment $e^{-rt}(V_D + \phi(F_D - F_0))$. So, $V_0 = e^{-rt}(V_D + \phi(F_D - F_0))$. We now solve for the price of V_0 :

$$V_0 = e^{-rt}(V_D + \phi(F_D - F_0))$$

$$= e^{-rT} \left(V_D + \frac{V_U - V_D}{F_D - F_U} (F_D - F_0) \right)$$

$$= e^{-rT} \left(V_D + (V_U - V_D) \left(\frac{F_0 - F_D}{F_U - F_D} \right) \right)$$

So $\frac{F_0 - F_D}{F_U - F_D}$ is the q of our earlier equation.

The condition $0 \leq q \leq 1$ reduces to $F_D \leq F_0 \leq F_U$ if $F_D \leq F_U$ and to $F_U \leq F_0 \leq F_D$ if $F_U \leq F_D$, which must be true to avoid arbitrage.

Note: If V is a forward instrument that does not have to be paid for until the end of time

T , then $V_0 = V_D + (V_U - V_D) \left(\frac{F_0 - F_D}{F_U - F_D} \right)$ without the discounting factor e^{-rT} .

Questions:

1. How are our conclusions affected if
 - a. $V_U = V_D$
 - b. $F_U = F_D$
2. Does it matter whether the value of F increases when V increases ($F_U > F_D$) or the value of F decreases when V increases ($F_D > F_U$)?
3. Does it matter whether the payoffs are positive or negative?
4. Does the relative size of changes in F and V matter, e.g. would the theory be any different if $(V_U - V_D)$ is very large and $(F_U - F_D)$ very small or if $(V_U - V_D)$ is very small and $(F_U - F_D)$ very large?
5. Have the probabilities of state U occurring rather than state D been involved anywhere in these calculations?
6. What would happen if we could decide on our hedge ratio after knowing whether prices will rise or fall (i.e., $\emptyset_U \neq \emptyset_D$)?
7. What if we choose $V = F$?

$$V_0 = (V_D + (V_U - V_D) \left(\frac{F_0 - F_D}{F_U - F_D} \right))$$

$$= (F_D + (F_U - F_D) \left(\frac{F_0 - F_D}{F_U - F_D} \right))$$

$$= (F_D + F_0 - F_D)$$

$$= F_0$$

So our formula gives a consistent result.

Now it certainly can be objected that it is unrealistic to assume that only 2 future states of the world are possible at a given time T . But, as we'll see next week, we can extend the results we've derived today to any sequence of binomial branches. So if we divide the time between now and T into enough different segments, we can cover as many different states of the world as we like at time T . We'll see, for example, that we can approximate a lognormal distribution of prices as closely as we like with a binomial branching process. But we still could object that a lognormal distribution could be produced in many different ways — it could result from a trinomial branching process just as easily as from a binomial branching process. And a trinomial branching process requires two hedging instruments, not just one, to achieve constant results at all branches. And a process with n -branches requires $n-1$ hedging instruments. In the continuous time theory, we'll find a more universal approach to solving this problem — by focusing on processes which are Brownian motions, we will utilize the Ito calculus to derive a single-instrument hedging strategy.