

## Derivative Securities -- Spring 2007 — Section 5

Notes by Robert V. Kohn, modified by Steve Allen  
Courant Institute of Mathematical Sciences

**The Black Scholes formula and its applications.** This section deduces the Black-Scholes formula for a European call or put, as a consequence of risk-neutral valuation in the continuous time limit. Then we discuss the Delta, Gamma, Vega, and Rho of a portfolio and their significance for hedging. Hedging is a very important topic, and these notes don't do justice to it; see, for example, chapter 15 of Hull's 6<sup>th</sup> edition for further, more practical discussion.

**The Black-Scholes formula for a European call or put.** The upshot of Section 4 is this: the value at time  $t$  to a European option with payoff  $V(F_T)$  is

$$V_0 = e^{-r(T-t)} E_{RN} [V(F_T)]$$

Here  $E_{RN} [V(F_T)]$  is the expected value of the price at maturity with respect to a special probability distribution – the risk-neutral one. This distribution is determined by the property that

$$F_T = F_t \exp \left[ -\frac{1}{2} \sigma^2 (T-t) + \sigma \sqrt{T-t} Z \right]$$

where  $F_t$  is the spot price at time  $t$  and  $Z$  is Gaussian with mean 0 and variance 1.

Equivalently,  $\log[F_T / F_t]$  is Gaussian with mean  $-\frac{1}{2} \sigma^2 (T-t)$  and variance  $\sigma^2 (T-t)$ .

This formula can be evaluated for any payoff  $V$  by numerical integration. But for special payoffs – including the put and the call – we can get explicit expressions in terms of the “cumulative distribution function”

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du$$

( $N(x)$  is the probability that a Gaussian random variable with mean 0 and variance 1 has value  $\leq x$ .) The explicit formulas have advantages over numerical integration. Besides being easy to evaluate, they permit us to see quite directly how the value and the hedge portfolio depend on strike price, spot price, risk-free rate, and volatility.

It's sufficient, of course, to consider  $t = 0$ . Let

$c[F_0, T, K]$  = value at time 0 of a European call with strike  $K$   
and maturity  $T$  if the forward price is  $F_0$

$p[F_0, T, K]$  = value at time 0 of a European put with strike  $K$  and maturity  $T$  if the forward price is  $F_0$

The explicit formulas are:

$$c[F_0, T; K] = e^{-rT} [F_0 N(d_1) - KN(d_2)]$$

$$p[F_0, T; K] = e^{-rT} [KN(-d_2) - F_0 N(-d_1)]$$

in which

$$d_1 = \frac{1}{\sigma\sqrt{T}} \left[ \log(F_0 / K) + \frac{1}{2} \sigma^2 T \right]$$

$$d_2 = \frac{1}{\sigma\sqrt{T}} \left[ \log(F_0 / K) - \frac{1}{2} \sigma^2 T \right] = d_1 - \sigma\sqrt{T}$$

To derive these formulas, we use the following result. (The statement at the end of Section 4 was a special case.)

**Lemma:** Suppose  $X$  is Gaussian with mean  $\mu$  and variance  $\sigma^2$ . Then for any real numbers  $a$  and  $k$ ,

$$E[e^{aX} \text{ restricted to } X \geq k] = e^{a\mu + \frac{1}{2}a^2\sigma^2} N(d)$$

with  $d = (-k + \mu + a\sigma^2) / \sigma$

**Proof:** The left hand side is defined by

$$E[e^{aX} \text{ restricted to } X \geq k] = \frac{1}{\sigma\sqrt{2\pi}} \int_k^\infty e^{ax} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx$$

Complete the square:

$$ax - \frac{(x-\mu)^2}{2\sigma^2} = a\mu + \frac{1}{2}a^2\sigma^2 - \frac{[x - (\mu + a\sigma^2)]^2}{2\sigma^2}.$$

Thus

$$E[e^{aX} | X \geq k] = e^{a\mu + \frac{1}{2}a^2\sigma^2} \cdot \frac{1}{\sigma\sqrt{2\pi}} \int_k^\infty \exp\left[-\frac{[x - (\mu + a\sigma^2)]^2}{2\sigma^2}\right] dx$$

If we set  $u = [x - (\mu + a\sigma^2)] / \sigma$  and  $\kappa = [k - (\mu + a\sigma^2)] / \sigma$  this becomes

$$e^{a\mu + \frac{1}{2}a^2\mu^2} \cdot \frac{1}{\sqrt{2\pi}} \int_k^\infty e^{-u^2/2} du = e^{a\mu + \frac{1}{2}a^2\sigma^2} [1 - N(\kappa)]$$

$$= e^{a\mu + \frac{1}{2}a^2\sigma^2} N(d)$$

where  $d = -\kappa = [-k + \mu + a\sigma^2] / \sigma$ .

Corollary 1 to Lemma.

For  $a = 0$ ,  $E(\exp(1, \text{restricted to } X \geq k)) = \text{Prob}(X \geq k) = N\left(\frac{\mu - k}{\sigma}\right)$

This can be shown independently of the lemma by

$$\text{Prob}(X \geq k) = \text{Prob}(X' \geq \frac{k - \mu}{\sigma})$$

(for  $X' = \frac{X - \mu}{\sigma}$  which has mean of 0 and standard deviation of 1)

$$= \text{Prob}(X' \leq \frac{\mu - k}{\sigma}) \text{ (by symmetry of the normal distribution)}$$

$$= N\left(\frac{\mu - k}{\sigma}\right)$$

Corollary 2 to Lemma

For  $a = 1$ ,  $E(\exp(X), \text{restricted to } X \geq k) = \exp(\mu + \frac{1}{2}\sigma^2) N\left(\frac{\mu - k}{\sigma} + \sigma\right)$

Proof of Theorem

$$\text{Let } X = \left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z$$

$$\text{Let } k = \log(K / S_0)$$

$$S_0 \exp(X) > K \text{ iff } X > k$$

So,  $E[(S_0 \exp(X) - K)_+] = E(S_0 \exp(X) - K, \text{ restricted to } S_0 \exp(X) > K)$

$= E(S_0 \exp(X) - K, \text{ restricted to } X > k)$

$= S_0 E(\exp(X), \text{ restricted to } X > k) - KE(1, \text{ restricted to } X > k)$

$$= S_0 \left( \exp\left(\mu + \frac{1}{2}\sigma^2\right) N\left(\frac{\mu - k}{\sigma} + \sigma\right) \right) - KN\left(\frac{\mu - k}{\sigma}\right)$$

(the first expression by corollary 2, the second expression by corollary 1)

$\mu_Z = 0$  and  $\sigma_Z = 1$ , so  $\mu_X = \left(r - \frac{1}{2}\sigma^2\right)T$  and  $\sigma_X = \sigma\sqrt{T}$ , so the previous expression

$$= S_0 \exp(rT) N\left(\frac{\log\left(S_0/K\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - KN\left(\frac{\log\left(S_0/K\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

and so  $e^{-rt} E[(S_0 \exp(X) - K)_+] = S_0 N(d_1) - e^{-rt} KN(d_2)$

The formula for the value of a European put can be obtained similarly. Or – easier – we can derive it from the formula for a call, using put-call parity:

$$\begin{aligned} p[F_0, T; K] &= c[F_0, T; K] + e^{-rT} (K - F_0) \\ &= e^{-rT} (K[1 - N(d_2)] - F_0[1 - N(d_1)]) \\ &= e^{-rT} (KN(-d_2) - F_0(-N(d_1))) \end{aligned}$$

For options with maturity  $T$  and strike price  $K$ , the value at any time  $t$  is naturally  $c[F_t, T - t; K]$  for a call,  $p[F_t, T - t; K]$  for a put.

\*\*\*\*\*

**Hedging.** We know how to hedge in the discrete-time, multiperiod binomial tree setting. The payoff is replicated by a portfolio consisting of  $\Delta = \Delta(0, F_0)$  units of the forward chosen to have the same value as the derivative claim. At time  $\delta t$  forward price changes to  $F_{\delta t}$  and the value of the hedge portfolio changes by  $\Delta(F_{\delta t} - F_0)$ . The new value of the hedge portfolio is also the new value of the option, so

$$\Delta(0, F_0) = \frac{\text{Change in value of option from time 0 to } \Delta t}{\text{Change in value of forward from time 0 to } \Delta t}$$

The replication strategy requires a self-financing trade at every time step, adjusting the amount of forward in the portfolio to match the new value of  $\Delta$ .

In the real world, prices are not confined to a tree, and there are no well-defined time steps. We cannot trade continuously. So while we can pass to the continuous time limit for the value of the option, we must still trade at discrete times in our attempts to replicate it. Suppose, for simplicity, we trade at equally spaced times with interval  $\delta t$ . What to use for the initial hedge ratio  $\Delta$ ? Not being clairvoyant, we don't know the value of the forward at time  $\delta t$  so we can't use the formula given above. Instead we should use its continuous time limit:

$$\Delta(0, F_0) = \frac{\partial(\text{value of option})}{\partial(\text{value of the forward})}$$

There's a subtle point here: if the forward price changes continuously in time but we only rebalance at discretely chosen times  $j\delta t$ , then we only rebalance at discretely chosen times, then we cannot expect to replicate the option perfectly using self-financing trades. Put differently, if we maintain the principle that the value of the hedge portfolio is equal to that of the option at each time  $j\delta t$ , then our trades will no longer be self-financing. We will address this point soon, after developing the continuous-time Black-Scholes theory. We'll show then that (if transaction costs are ignored) the expected cost of replication tends to 0 as  $\delta t \rightarrow 0$ . (In practice, transaction costs are *not* negligible. Deciding when to rebalance, taking into account transaction costs, is an important and interesting problem – but one beyond the scope of this course.)

For the European put and call we can easily get formulas for  $\Delta$  by differentiating our expressions for  $c$  and  $p$ : at time  $T$  from maturity the hedge ration should be

$$\Delta = \frac{\partial}{\partial F_0} c[F_0, T; K] = e^{-rT} N(d_1)$$

for the call, and

$$\Delta = \frac{\partial}{\partial F_0} p[F_0, T; K] = e^{-rT} N(-d_1)$$

for the put. This is an application of the chain rule. For example, in the case of the call,

$$\frac{\partial}{\partial F_0} c = e^{-rT} N(d_1) + e^{-rT} F_0 N'(d_1) \frac{\partial d_1}{\partial F} - K e^{-rT} F_0 N'(d_2) \frac{\partial d_2}{\partial F}$$

But  $d_2 = d_1 - \sigma\sqrt{T}$ , so  $\partial d_1 / \partial F = \partial d_2 / \partial F$ ; also  $N'(x) = \frac{1}{\sqrt{2\pi}} \exp[-x^2/2]$ . It follows with

some calculation that  $e^{-rT} F_0 N'(d_1) \frac{\partial d_1}{\partial F} - K e^{-rT} F_0 N'(d_2) \frac{\partial d_2}{\partial F} = 0$  so finally

$$\frac{\partial c}{\partial F_0} = e^{-rT} N(d_1) \text{ as asserted.}$$

\*\*\*\*\*

Since we have explicit formulas for the value of a put or call, we can differentiate them to learn the dependence on the underlying parameters. Some of these derivatives have names:

Definition	Call	Put
Delta = $\frac{\partial}{\partial F_0}$	$e^{-rT} N(d_1) > 0$	$e^{-rT} N(-d_1) < 0$
Gamma = $\frac{\partial^2}{\partial F_0^2}$	$-\frac{1}{F_0 \sigma \sqrt{2\pi T}} \exp(-d_1^2/2) > 0$	$\frac{1}{F_0 \sigma \sqrt{2\pi T}} \exp(-d_1^2/2) > 0$
Theta = $\frac{\partial}{\partial t}$	$-\frac{F_0 \sigma}{2\sqrt{2\pi T}} \exp(-d_1^2/2) - r K e^{-rT} N(d_2) < 0$	$-\frac{F_0 \sigma}{2\sqrt{2\pi T}} \exp(-d_1^2/2) + r K e^{-rT} N(-d_2) > 0$
Vega = $\frac{\partial}{\partial \sigma}$	$-\frac{F_0 \sqrt{T}}{\sqrt{2\pi}} \exp(-d_1^2/2) > 0$	$-\frac{F_0 \sqrt{T}}{\sqrt{2\pi}} \exp(-d_1^2/2) > 0$
Rho = $\frac{\partial}{\partial r}$	$T K e^{-rT} N(d_2) > 0$	$-T K e^{-rT} N(d_2) < 0$

These formulas apply at time  $t = 0$ ; the formulas applicable at any time  $t$  are similar, with  $T$  replaced by  $T - t$ . These are obviously useful for understanding how the value of the option changes with time, volatility, etc. But more: they are useful for designing improved hedges. For example, suppose a bank sells two types of options on the same underlying asset, with different strike prices and maturities. As usual, the bank wants to limit its exposure to changes in the forward price; but suppose, in addition, it wants to limit its exposure to changes (or errors in the specification of) volatility. Let  $i = 1, 2$  refer to the two types of options, and let  $n_1, n_2$  be the quantities held of each. (These are negative if the bank sold the options.) The bank naturally also invests in the underlying

forward and in risk-free bonds; let  $n$  be the quantities held of each. Then the value of the bank's initial portfolio is

$$V_{total} = n_1 V_1 + n_2 V_2 + n_s F_0 + n_b$$

We already know how the forward and bond holdings should be chosen if the bank plans to replicate (dynamically) the options. They should satisfy

$$V_{total} = 0$$

and

$$N_1 \Delta_1 + n_2 \Delta_2 + n_s = 0$$

Notice that the latter relations says  $\partial V_{total} / \partial F_0 = 0$ : the value of the bank's holdings are insensitive (to first order) to changes in the forward price.

If we were dealing in just one option, there would be no further freedom. We would have two homogeneous equations in three variables  $n_1$ ,  $n_s$ ,  $n_b$ , restricting their values to a line – so that  $n_1$  determines  $n_s$  and  $n_b$ . That's the situation we're familiar with. But if we're dealing in two (independent) options, then we have the freedom to impose one additional linear equation. For example, we can ask that the portfolio be insensitive (to first order) to changes in  $\sigma$  by imposing the additional condition

$$n_1 Vega_1 + n_2 Vega_2 = 0.$$

Thus, by selling the two types of assets in the proper proportions, the bank can reduce its exposure to change or misspecification of volatility.

If the bank sells three types of options, then we have room for yet another condition, e.g., we could impose first-order insensitivity to changes in the risk-free rate  $r$ . And so on. It is not actually necessary that the bank use the underlying forward as one of its assets. Each option is *equivalent* to a portfolio consisting of forwards and risk-free bonds; so a portfolio consisting entirely of options and a bond position will function as a hedge portfolio so long as its total  $\Delta$  is equal to 0.

Replication requires dynamic rebalancing. The bank must change its holdings at each time increment to set the new  $\Delta$  to 0. In the familiar one-option setting, this was done by adjusting the forward and bond holdings, keeping the option holding fixed. In the present two-option setting, maintaining the additional condition  $Vega_{total} = 0$  will require the ratio between  $n_1$  and  $n_2$  to be dynamically updated as well, i.e., the bank will have to sell or buy additional options as time proceeds.