**Derivative Securities -- Spring 2007 — Section 7**

Notes by Robert V. Kohn, modified by Steve Allen
Courant Institute of Mathematical Sciences

## Martingales

The basic prescription for working backward in a binomial tree is this: if $V$ is the price of a tradeable security (such as an option) then

$$V_{now} = e^{-r\delta t}\,[qV_{up} + (1 - q)V_{down}] = e^{-r\delta t}\, E_{RN}[V_{next}]$$

and if $V$ is the futures price of a tradeable security then

$$V_{now} = [qV_{up} + (1 - q)\, V_{down}] = E_{RN}\, [V_{next}],$$

where $q$ is the risk-neutral probability, defined by

$$F_{now} = qF_{up} + (1 - q)F_{down} = E_{RN}\, [F_{next}].$$

When the risk-free rate is constant, the factors of $e^{-r\delta t}$ don't bother us — we just bring them out front. When the risk-free rate is stochastic, however, we must handle them differently. To this end it is convenient to introduce a *money market account* which earns interest at the risk-free rate. Let $A(t)$ be its balance, with $A(0) = 1$. In the constant interest rate setting, obviously $A(t) = e^{rt}$; in the variable interest rate setting, we still have $A(t + \delta t) = e^{r\delta t} A(t)$. However, $r$ might vary from time to time, and even (if interest rates are stochastic) from one binomial subtree to another. With this convention, the prescription for determining the price of a tradeable security becomes

$$V_{now} / A_{now} = E_{RN}\, [V_{next} / A_{next}]$$

since $A_{now} / A_{next} = e^{-r\delta t}$ where $r$ is the risk-free rate. (This relation is valid even if the risk-free rate varies from one subtree to the next). Working backward in the tree, this relation generalizes to one relating the option value at any pair of times $0 \le t < t' \le T$:

$$V(t) / A(t) = E_{RN}\, [\, V(t') / A(t')\, ].$$

Here, as usual, the risk-neutral expectation weights each state at time $t'$ by the probability of reaching it via a coin-flipping process starting from time $t$ — with independent, biased coins at each node of the tree, corresponding to the risk-neutral probabilities of the associated subtrees.

The preceding results say, in essence, that certain processes are *martingales*. Concentrating on binomial trees, a "process" is just a function *g* whose values are defined at every node. A process is said to be a *martingale* relative to the risk-neutral probabilities if it satisfies

$$g(t) = E_{RN}[g(t')]$$

for all $t < t'$. The risk-neutral probabilities are determined by the fact that

- $F(t)$ is a martingale relative to the risk-neutral probabilities

where $F(t)$ is the futures price process. Option prices are determined by the fact that

- $V(t) / A(t)$ is a martingale relative to the risk-neutral probabilities

if $V$ is the price of a tradeable asset. Prices of other futures are determined by the fact that

- $V(t)$ is a martingale relative to the risk-neutral probabilities.

One advantage of this framework is that it makes easy contact with the continuous-time theory. The central connection is this: in continuous time, the solution of a stochastic differential equation $dv = \mu dt + \sigma dz$ is a martingale if $\mu = 0$. Indeed, the expected value of a $dz$-stochastic integral is 0, since Brownian motion by definition has mean value 0, so for any $t < t'$ we have $E[y(t)] = E\left[\int_t^{t'} \mu(\tau)d\tau\right] = \int_t^{t'} E[\mu(\tau)]d\tau$ for the right hand side to vanish (for all $t < t'$) we must have $E[\mu] = 0$. If $\mu$ is deterministic then this condition says simply that
$\mu = 0$.

We can use this insight to give a more intuitive derivation of our fundamental equation from the Balck-Scholes PDE. We return here to the constant-interest-rate environment, so $A(t) = e^{rt}$.

If a derivative satisfies the Black-Scholes PDE, it is a martingale under the risk-neutral probability distributon in which $dF = \sigma F dz$, with a zero drift term. Suppose the option price has the form $V(F(t), t)$ for some function $V(F, t)$. Then

$$\begin{aligned} d\left(V(F(t),t)e^{-rt}\right) &= e^{-rt}dV - re^{-rt}Vdt \\ &= e^{-rt}(V_t dt + V_F dF + \tfrac{1}{2}V_{FF}\sigma^2 F^2 dt) - re^{-rt}Vdt \qquad \text{(applying Ito's lemma to dV)} \\ &= e^{-rt}(V_t + \tfrac{1}{2}\sigma^2 F^2 V_{FF} - rV)dt + e^{-rt}\sigma F V_F dz \qquad \text{(using } dF = \sigma F dz\text{)} \end{aligned}$$

For this to be a martingale the coefficient of $dt$ must vanish. That is exactly the Black-Scholes PDE.

The solution of the Black-Scholes PDE gives the discounted expected payoff of the option. Suppose $V$ solves the Black-Scholes PDE, with final value $V_T(F_T)$ We have shown that $e^{-rt}V_t(F_t,)$ is a martingale relative to the risk-neutral probability distribution (note that we're switching notation here to use $V_t$ to mean the value of V at time t, rather than the derivative of V with respect to t, as in the last block of equations). Therefore

$$V_0(F_0) = E_{\mathrm{RN}}[e^{-rt}V_t(F_t)]$$

for any $t > 0$. Bringing $e^{-rt}$ out of the expectation and setting $t = T$ gives

$$V_0(F_0) = e^{-rT}\, E_{\mathrm{RN}}[V_T(F_T)]$$

as asserted.

**[The remainder of this section will closely parallel the section 1 notes for Continuous Time Finance]**