Derivative Securities, Courant Institute, Fall 2006
Assignment 3, due October 4

Important: Always check the class bboard on the blackboard site from home.nyu.edu (click on academics, then on Derivative Securities) before doing any work on the assignment. In particular, some questions may be deleted or added depending on how much material we cover in class.

1. Let X be a Gaussian random variable with mean zero and variance σ^2.

   (a) Show that for any positive integer, n,
   $$E\left[X^{2n}\right] = \sigma^{2n}(2n-1)(2n-3)\cdots(3) \;. \tag{1}$$
   Hint: derive the *recurrence relation*
   $$E\left[X^{2n}\right] = \sigma^2(2n-1)E\left[X^{2n-2}\right]$$
   by integration by parts using $xe^{-x^2/2\sigma^2} = -\sigma^2\partial_x e^{-x^2/2\sigma^2}$ and $x^{2n} = x^{2n-1}\cdot x$.

   (b) Use the Laplace method from class to find an approximate formula like Stirling's formula for $E\left[X^{2n}\right]$. This involves finding x_* to maximize the integrand, writing the integrand as an exponential, then finding a quadratic approximation to the exponent that gives a Gaussian approximation to the integral. There are two maximizing values, but their structure is similar.

   (c) Make a plot in a neighborhood of one of the x_* points that shows the actual integrand and the Gaussian approximation to it. Do this for $n = 2, 3, 4, 5, ...$ You will see how the Gaussian approximation starts to work better for larger n. You also will see that for larger n, most of the area under the curve is near one of the x_*.

   (d) Compare the approximate answer to the exact answer for $n = 2, 3, 4$.

   (e) *Discussion*: One purpose of this exercise is to give you practice with Gaussian random variables and to think through the ideas of the Laplace method. This method has several applications in finance, including estimating the probabilities of rare events by finding the most likely way an event of a certain kind can happen, which is useful in risk assessment.

   Moreover, the moment formulas (1) characterize Gaussian random variables. If X is a random variable that satisfies (1) for all n, then X is Gaussian. If X is a general random variable with $E[X] = 0$ and $\text{var}[X] = \sigma^2$, the *skewness*, $E[X^3]/\sigma^3$, and *kurtosis*, $E[X^4]/\sigma^4 - 3$, quantify the extent to which X is not Gaussian. For example, daily log stock returns (which are Gaussian in the Black Scholes theory) have considerable skewness and kurtosis.

2. The *geometric Brownian motion* is given by

$$S(t) = S_0 \exp\left(\sigma W(t) + \left(\mu - \frac{\sigma^2}{2}\right) t\right) .$$

Here, $W(t)$ is a *standard* Brownian motion (or *Wiener process*, after Norbert Wiener who helped define it). Of the many properties of Brownian motion, the only one we need here is that $W(t)$ is a Gaussian random variable with $E[W(t)] = 0$ and $\text{var}[W(t)] = t$. The parameters are μ, the rate of return, and σ, the volatility.

(a) Let X be a Gaussian random variable with $E[X] = \mu$ and $\text{var}[X] = \sigma^2$. Calculate $E[e^X]$. Hint: Write the integral defining the expected value, then complete the square in the exponent of the integrand. Some find it simpler to use a standard normal, Z, which is a Gaussian with $E[Z] = 0$ and $\text{var}[Z] = 1$. Then $X = \sigma Z + \mu$ is Gaussian with mean μ and variance σ^2, so we may instead calculate $E[e^{\sigma Z + \mu}]$ for a standard normal Z.

(b) Show that $E[S(t)] = S_0 e^{\mu t}$.

(c) *Discussion* The manipulations that answer part (a) are a part of the derivation of the Black Scholes formula, which is the discounted expected payout of a European put or call option under geometric Brownian motion with μ replaced by r. If $\mu = 0$, the geometric Brownian motion is a *martingale*, which means that its expected value does not change with time. Part (b) shows this.