

Derivative Securities, Courant Institute, Fall 2006
 Assignment 4, due October 11 [Update: typo fixed 10/6][Update
 10/9: Formulas in 1a fixed, 1e expanded.]

Important: Always check the class bboard on the blackboard site from home.nyu.edu (click on academics, then on Derivative Securities) before doing any work on the assignment. In particular, some questions may be deleted or added depending on how much material we cover in class.

1. Let S_k be a binomial tree process with time period Δt , volatility σ and expected growth rate μ . Suppose that $S_{k+1} = M_k S_k$, where M_k is independent of S_j for $j \leq k$ and

$$\Pr(M_k = u) = P_u, \quad \Pr(M_k = d) = P_d = 1 - P_u, \quad P_u = P_d = \frac{1}{2}. \quad (1)$$

and

$$u = e^{\sigma\sqrt{\Delta t} + (\mu - \frac{\sigma^2}{2})\Delta t}, \quad d = e^{-\sigma\sqrt{\Delta t} + (\mu - \frac{\sigma^2}{2})\Delta t}. \quad (2)$$

Use the Taylor expansion

$$e^\epsilon = 1 + \epsilon + \frac{1}{2}\epsilon^2 + O(\epsilon^3). \quad (3)$$

The step, or *shock*, at period k is $\Delta S_k = S_{k+1} - S_k$.

- (a) Verify the relations

$$\begin{aligned} E[\Delta S_k | \mathcal{F}_k] &= \mu S_k \Delta t + O(\Delta t^2) \\ E[\Delta S_k^2 | \mathcal{F}_k] &= \sigma^2 S_k^2 \Delta t + O(\Delta t^2) \\ \text{var}[\Delta S_k | \mathcal{F}_k] &= \sigma^2 S_k^2 \Delta t + O(\Delta t^2) \\ E[\Delta S_k^4 | \mathcal{F}_k] &= O(\Delta t^2) \end{aligned}$$

Here \mathcal{F}_k means the values S_0, S_1, \dots, S_k .

- (b) Suppose we replace μ by r in (2). What values of P_u and P_d in (1) are needed to make the relations in part (a) hold? Note that P_u and P_d only are defined up to $O(\Delta t^2)$. Choose simple formulas for P_u and P_d that work.
- (c) Calculate the risk neutral probabilities, P_u^{RN} and P_d^{RN} , for the binomial tree given by (3). Show that these lead to the formulas in part (a) with μ replaced by r . Your formula should involve the expression $(\mu - r)/\sigma$.
- (d) Argue from this, and the convergence theorem from class, that the risk neutral process in the continuous time limit, $\Delta t \rightarrow 0$, $N \rightarrow \infty$ with $T = N\Delta t$ fixed, satisfies the SDE $dS = rSdt + \sigma SdW$.

- (e) *Discussion:* There are two themes here. One is that the $O(\epsilon^2)$ term in (2), applied to $\sqrt{\Delta t}$ gives a result of order Δt and therefore crucial for getting the right answer. The other is a different way to see that the relation between the actual model of the stock price process, $dS = \mu S dt + \sigma S dW$, and the risk neutral process with μ replaced by r . In this direction, note that the answers to parts (b) and (c) are the same.
2. Ordinary Brownian motion is characterized by (using the notation $\Delta W = W(t + \Delta t) - W(t)$, and \mathcal{F}_t for all the values $W(s)$ for $0 \leq s \leq t$):

$$\begin{aligned} W(0) &= 0 \\ E[\Delta W \mid \mathcal{F}_t] &= 0 \\ E[(\Delta W)^2 \mid \mathcal{F}_t] &= \Delta t \\ E[(\Delta W)^4 \mid \mathcal{F}_t] &= 3\Delta t^2 \end{aligned}$$

Use this, together with Ito's lemma from class, to show that the function

$$S(t) = S_0 e^{\sigma W(t) + (\mu - \frac{\sigma^2}{2})t} \quad (4)$$

satisfies the SDE $dS = \mu S dt + \sigma S dW$. *Discussion:* Solving a differential equation or calculating an integral largely is a matter of systematic guessing then checking through differentiation. The same is true for SDEs, with Ito's lemma playing the role of the differentiation formula.

3. The cumulative normal is $N(x) = \Pr(Z \leq x)$, where Z is a mean zero variance one Gaussian random variable. For simplicity, look at the complementary function

$$M(x) = \Pr(Z > x) = \frac{1}{\sqrt{2\pi}} \int_{z=x}^{\infty} e^{-z^2/2} dz .$$

The change of variable $z = x + y$ puts this in the form

$$M(x) = \frac{e^{-x^2}}{\sqrt{2\pi}} \int_{y=0}^{\infty} e^{-xy} e^{-y^2/2} dy .$$

For large x , e^{-xy} goes to zero so rapidly that $e^{-y^2/2}$ is essentially constant.

- (a) Make plots of the two functions $e^{-xy} e^{-y^2/2}$ and e^{-xy} as functions of y for $x = 2, 5, 20$ to verify this.
- (b) Using the approximation $e^{-y^2/2} \approx 1$, find an explicit approximate formula

$$M(x) \approx \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2} .$$

Use this to write the large x approximations:

$$N(x) \approx ?? \quad \text{as } x \rightarrow -\infty, \quad (5)$$

$$N(x) \approx 1 - ?? \quad \text{as } x \rightarrow \infty. \quad (6)$$

- (c) Find an approximate formula for a far out of the money put or call that uses (5). You need to use the Black Scholes formula for a vanilla European put or call on a non dividend bearing stock, which is in the Kohn and Allen notes and the Hull book.
 - (d) Find an approximate formula for a deep in the money put or call, either using (6) or the result of part (c) together with put/call parity.
 - (e) Note that the answer to part (d) is very close to the value of a related forward contract. Why?
 - (f) *Discussion:* Approximate formulas of all kinds are very useful as reality checks for analytic or numerical pricers. However, the Black Scholes pricing theory for far out of the money options is questionable. In general, such options sell for much more than the at the money prices would suggest. One theory is that this is because the supply, particularly of out of the money puts as portfolio protection, is less than the demand.
4. An interesting financial instrument is a forward struck call. In this option, we set $K = S(T_1)$ and receive the call option for this strike price at time $T_2 > T_1$. Use the Black Scholes theory to find a closed form formula for the price of such an option as a function of S, t, σ , and r . Hint: you will need things like $\int_{-\infty}^x N(y)dy$. You can express this in terms of exponentials and $N(x)$ itself by writing $N(y)$ as an integral, and $\int_{-\infty}^x N(y)dy$ as a two dimensional integral then changing the order of integration. *Discussion:* The Black Scholes theory applies to many options other than vanilla puts and calls. The next assignment will feature digital options and options on the maximum of two stocks. The integral of $N(y)$ will come in handy when we discuss the shape of the option price curve for short time. Most equity options are for short time, while options for currencies (FX), interest rates, and credit can be for much longer.