Derivative Securities, Courant Institute, Fall 2006
Assignment 5, due October 25

Important: Always check the class bboard on the blackboard site from home.nyu.edu (click on academics, then on Derivative Securities) before doing any work on the assignment. In particular, some questions may be deleted or added depending on how much material we cover in class.

1. Suppose $S_1(t)$ and $S_2(t)$ are independent geometric Brownian motions with growth rates μ_1 and μ_2, and volatilities σ_1 and σ_2 respectively. We own S_1 and have the option at time T to trade S_1 for S_2 What is the Black Scholes price for this option?

   (a) Let $f(s_1, s_2, t)$ be the price for this option at tiem $t < T$ with prices $S_1(t) = s_1$ and $S_2(t) = s_2$. Find values of Δ_1 and Δ_2 so that the portfolio $\Pi = f - \Delta_1 S_1 - \Delta_2 S_2$ is "infinitesimally" risk free. That is, $d\Pi$ has to have no dS_1 or dS_2 component.

   (b) Show that
   $$f(s_1, s_2, 0) = e^{-rT} \mathrm{E}^{RN} \left[(S_2 - S_1)_+ \right] \ ,$$
   and identify the risk neutral processes for S_1 and S_2.

   (c) Express the right side in part (b) as a sum of two double integrals, then calcuate one integral of each term to express the result as the sum of two single integrals, each involving the cumulative normal distribution function $N(z)$. Hint: Express the risk neutral $S_1(T)$ and $S_2(T)$ as exponentials involving standard normals Z_1 and Z_2 as we did in deriving the Black Scholes formula in class.

   (d) Suppose $Y \sim \mathcal{N}(0, \sigma^2)$, express $\Pr(Y \leq c)$ in terms $N(z)$. We can define $N(z) = \Pr(Z \leq z)$ where $Z \sim \mathcal{N}(0, 1)$ is a standard normal.

   (e) Evaluate
   $$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} N(az_1 + b) e^{-z_1^2/2} dz_1 \ .$$
   Hint: $N(az_1 + b) = \Pr(Z_2 \leq az_1 + b)$ if $Z_2 \sim \mathcal{N}(0, 1)$. The integral is
   $$E_{Z_1} \left[\Pr\left(Z_2 \leq aZ_1 + b \right) \right] \ ,$$
   where $E_{Z_1}[\cdot]$ means to take the expected value (integrate) over z_1 and $Z_1 \sim \mathcal{N}(0, 1)$ is independent of Z_2. This expectation is $\Pr(Z_1 - aZ_2 \leq b)$. Now use part (d).

   (f) Use the result of part (e) to calculate both the integrals in part (c) and get a formula for $f(s_1, s_2, 0)$ in terms of N functions.

   (g) *Discussion*: This is a math problem that gives you practice working with exponentials, etc. The interplay between calculus and probability, finding probabilistic interpretations of integral expressions as in parts (d) and (e), makes calculations easier and more intuitive.

2. Suppose $t_1 < t_2 < t_3$ are times and $X_1 = X(t_1)$, $X_2 = X(t_2)$, and $X_3 = X(t_3)$ are the values of a diffusion process at these times. Define

$$f(x,t) = \mathrm{E}_{x,t}\left[V(X(t_3))\right] = \mathrm{E}_{x,t}\left[V(X_3)\right] \;,$$

where $\mathrm{E}_{x,t}[\cdot]$ means the expected value given that $X(t) = x$. If $X(t)$ satisfies $dX(t) = a(X(t))dt + b(X(t))dW(t)$, then there is the *transition probability*, or *transition density*, $G(x, y, \Delta t)$, which is the probability density for $X(t + \Delta t)$ given that $X(t) = x$. The transition density does not depend on t, only on Δt, because $a(x)$ and $b(x)$ do not depend on t. The *Markov property* (which holds for time independent a and b) is that these transition densities determine everything about the process $X(t)$. For example, let $H(x_1, x_2, x_3, t_1, t_2, t_3)$ be the joint density of $X_2 = X(t_2)$ and $X_3 = X(t_3)$ given that $X(t_1) = x_1$. This satisfies

$$H(x_1, x_2, x_3, t_1, t_2, t_3) = G(x_1, x_2, t_2 - t_1) \cdot G(x_2, x_3, t_3 - t_2) \;. \qquad (1)$$

(a) Verify the version of the *tower property* we used in class, namely

$$f(x_1, t_1) = \mathrm{E}_{x_1,t_1}\left[f(x_2, t_2)\right] \qquad (2)$$

using (1). Hint: Write (and explain why this is correct)

$$f(x_1, t_1) = \int\int V(x_3) H(x_1, x_2, x_3, t_1, t_2, t_3) dx_2 dx_3 \;,$$

then integrate over x_3 using (1).

(b) Suppose

$$dX = -\alpha dt + \sigma dW \qquad (3)$$

(Brownian motion plus drift). Find the formula for G. You can do this without writing or solving a partial differential equation. Just say what the the solution of (3) is (explaining why) and use it to determine the probability density.

(c) Verify (2) for Brownian motion with drift by explicitly evaluating the integrals on the right and left side of (2) for $V(x) = x^2$.

(d) Show that $f(x,t)$ satisfies the backward equation for this process by explicitly differentiating it.

(e) *Discussion*: The formula (2) is central to our understanding of stochastic processes in general, and to the derivation of the backward equation in particular. There are not very many examples that can be calculated explicitly as here, and most of those are more complicated than this example.