

Derivative Securities, Courant Institute, Fall 2006  
Assignment 6, due November 1

**Important:** Always check the class bboard on the blackboard site from home.nyu.edu (click on academics, then on Derivative Securities) before doing any work on the assignment. In particular, some questions may be deleted or added depending on how much material we cover in class.

1. We explore some aspects of the log variable transformation.

- (a) Let  $f(s, t) = e^{rt}g(\log(s), t)$  and find the PDE  $g$  satisfies if

$$\partial_t + \frac{\sigma^2 S^2}{2} \partial_s^2 f + rS \partial_s f - rf = 0.$$

- (b) Find the final data,  $g(x, T) = W(x)$  consistent with the European put values  $f(s, T) = V(s) = (K - s)_+$ .
- (c) What is the diffusion process whose backward equation is the  $g$  PDE from part (a)?
- (d) What is the transition probability density,  $G(x_0, x_1, T)$ , for the solution to the  $X$  process from part (c)?
- (e) Use the formula for  $G$  from part (d) and the final values  $W(x)$  from part (b) to write an integral expression for  $g(x, 0) = E_{x,0}[W(X(T))]$ . Show that this is the same integral we used to find the Black Scholes formula for the price of that European style put.
- (f) Show that if  $dS = rSdt + \sigma SdW$ , and  $X(t) = \log(S(t))$ , then  $X$  satisfies the SDE from part (c).
- (g) *Discussion:* The log variable transformation is a popular way to derive the Black Scholes formula from the Black Scholes PDE. You should know it. You can do the log variable transformation using either the PDE as in part (a), or the SDE as in parts (d) and (e). A little practice with Ito's lemma, part (f), always is worthwhile.
2. Use a spreadsheet program (Microsoft Excel or OpenOffice or something with comparable functionality) to program the Black Scholes formula. A simple way to do this is to put the parameters  $r$ ,  $\sigma$ ,  $T$ ,  $K$ , and  $S$  into cells A1, A2, A3, A4, and A5 respectively, then put  $d_1$  and  $d_2$  into cells A6 and A7, then assemble the put value into cell A8 and the call value into cell A9. In the present assignment we will take  $r = 3\%$  always, but we will use other  $r$  values in future assignments.<sup>1</sup> As a matter of personal honor, each person must write her or his own Black Scholes formula. Any person

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<sup>1</sup>As a matter of programming practice, it usually is a bad idea to *hard wire* values of constants like  $r$ , even if we don't think the value will change. For one thing, it might change. For another, this assembles all the constants in one place so someone reading the spreadsheet can see what you're doing. For a third, if you use the numerical value  $.03 = 3\%$  in several places, it is easy to get it wrong in one of them.

who uses a copied Black Scholes spreadsheet must commit Klingon ritual suicide (ask a Star Trek fan what this means). Use this (and appropriate modifications) to carry out the following. Be sure to report numbers with enough significant figures to make the answers to the questions clear. Remember that 1% of a penny can be a significant amount of money, if you own enough shares or trade often enough.

- (a) What is the value of an at the money put with  $S(0) = K = \$100$  with  $\sigma = 30\%$ /year that expires in 3 months?
- (b) What is the value of a call with the same parameters? Check that these satisfy put call parity with part (a).
- (c) What is the value of the put from part (a) but struck at  $K = \$120$ ? Compare this to the cash and carry price of a forward contract with the same  $K$  and  $S(0)$ .
- (d) What is the value of the call from part (b) but struck at  $K = \$80$ ? Compare this to the cash and carry price of a forward contract with the same  $K$  and  $S(0)$ .
- (e) What is the value of the put from part (a) but struck at  $K = \$80$ ?
- (f) What are the probabilities that the put from part (c) will expire out of the money and that the put from part (e) will expire in the money. Comment on how these probabilities are consistent with the numbers from part (c) and part (e).
- (g) Consider a put with  $K = \$100$ ,  $T = 5$  years, and  $\sigma = 10\%$ /year. Make a plot of the price of the put as a function of the spot price at time  $t = 0$  running from  $s = \$20$  to  $s = \$120$ . On the same plot, indicate (make the computer indicate) the *intrinsic value*,  $(K - s)_+$ . The intrinsic value is what you would get if you allowed to exercise the put immediately rather than having to wait 5 years. Note (by eye, not by solving equations) where the price curve crosses the intrinsic value curve. This is the point at which you would prefer immediate exercise. Note the range of values where  $f(s, 0)$  has significant convexity. Explain the width of this region and why it is not centered about  $K$ .
- (h) Carry out the operations of part (g) for a call option. Does it intersect its intrinsic value curve?
- (i) *Discussion:* I want you to do your own spreadsheet programming because it is an important skill and because I want to rub in the Black Scholes formula. For part (a), I just want you to get an idea of the order of magnitude of an option price in a typical situation. Part (b) is more of generic option pricing and a check that you've programmed the formulas correctly. It is important to test every single line of code somehow before trusting it. Parts (c) and (d) are about understanding in the money options. It's also another check of the coding. Parts (d) and (e) and (f) show that far out of the

money options are worth very little in the Black Scholes world. Out of the money puts often are worth more in the marketplace than Black Scholes theory would suggest.

Part (g) is partly an exercise in working with spreadsheets. You have to figure out how to apply the Black Scholes formula to a long list of spot prices and plot the results. Putting all the numbers in the Black Scholes calculation in row A makes it possible to make fifty copies by dragging (figure out how to do this). You can create 50  $s$  values easily. Please don't do it manually. Please do not type in 50 prices by hand and apply the formula one by one. Make the spreadsheet do the typing. The graph shows that the price goes below the intrinsic value for a put (but not for a call, part (h)). This is what makes American style options that allow the holder to exercise early more valuable than European options that do not.