Derivative Securities, Courant Institute, Fall 2006
Assignment 9, due December 13

Important: Always check the class bboard on the blackboard site from home.nyu.edu
(click on academics, then on Derivative Securities) before doing any work on the
assignment. In particular, some questions may be deleted or added depending
on how much material we cover in class.

Important: There will be another assignment next week, so don't sit on this
one. It make take some time to get the Excel part working if you're not an
expert.

1. This exercise leads to a Δ, Γ, and Λ hedge of an *down-and-out* put on an
   equity index using the underlying index future and two vanilla puts on
   the index. In the process, we will use more features of Excel. Most of the
   Excel features used in this exercise are described in

   `http://faculty.fuqua.duke.edu/~pecklund/ExcelReview/ExcelReview.htm`

   (a) Start a new spreadsheet and name it `BarHedge`. Put the main param-
       eters in the top left, with names in column `A` and values in column
       `B`. Give every parameter a name (`S0` for S_0, etc.) so you can access
       it in formulas by name rather than by cell location. For example,
       the formula for the Black Scholes Γ of the first vanilla put will be `=`
       `BSG(S0, rf, q, T1, K1, Sig1)`. The parameters are:
       - Market prices
         – S_0 (spot index level),
         – f_1 (price of first vanilla put, if given)
         – f_2 (price of second vanilla put, if given)
       - Market parameters
         – r (risk free rate for the market)
         – q (dividend yield for the index)
         – σ_b (volatility, if specified externally)
       - Parameters for down-and-out put
         – K_b (strike price)
         – S_b (knockout price)
         – T_b (expiration time in years)
       - First vanilla put for hedging
         – K_1 (strike price)
         – T_1 (expiration time in years)
       - Second vanilla put for hedging
         – K_2 (strike price)
         – T_2 (expiration time in years)

In columns C and D put results from the Black Scholes formulas, with labels in column C and values in column D as before. Values here are copied or calculated from numbers in column B. Include:

- Futures price (**theoretical!**)
  - F_0 (price),
  - Δ_0 (Delta of F_0)
  - Γ_0 (Gamma of F_0)
  - Λ_0 (Vega of F_0)
- First vanilla put)
  - f_1 (price, theoretical or market),
  - σ_1 (vol used for f_1)
  - Δ_1 (Delta of f_1)
  - Γ_1 (Gamma of f_1)
  - Λ_1 (Vega of f_1)
- Second vanilla put)
  - f_2 (price, theoretical or market),
  - σ_2 (vol used for f_2)
  - Δ_2 (Delta of f_2)
  - Γ_2 (Gamma of f_2)
  - Λ_2 (Vega of f_2)

(b) Modify the `ForwardEuler` macro from Homework 7 to be a subroutine that values a European down and out put. This is the same as putting in the boundary condition $f = 0$ at the knockout boundary $s = S_b$. It should get its parameters from the appropriate places in column B and write the results in the appropriate places in column D. Make sure to refer to these parameters and computed results by name, not cell location. You need to modify the trinomial tree so that at time $t = 0$ there are three values, $f(s_0, 0)$, $f(s_0 - \Delta s, 0)$, and $f(s_0 + \Delta s, 0)$. These three values allow you to estimate Δ and Γ. Create appropriate places in column D to put the answers.

(c) Modify the subroutine again to compute $\Lambda = \partial_\sigma f(s, t)$ using the forward Euler method. The trick here is to differentiate the Black Scholes PDE with respect to σ and write the PDE satisfied by $\Lambda(s, t) = \partial_\sigma f(s, t)$. The final condition is $\Lambda(s, T) = \partial_\sigma f(s, T) = \partial_\sigma V(s) = 0$. The knockout boundary condition is $\Lambda(S_b, t) = 0$. Again, the answer goes back into the spreadsheet.

(d) Do some checks of the macros from part b and part c. If the knockout boundary is far enough away, the exotic option will have the same price as the vanilla and you can check Δ and Γ against the Black Scholes formulas. If the knockout boundary is closer, you can run the macro by hand with different s_0 and σ and see that the semi-analytical results from the forward Euler computation match straightforward finite difference estimates.

(e) A hedged portfolio, $\Pi = f_0 - w_0 F_0 - w_1 f_1 - w_2 f_2$, has its own Δ, Γ, and Λ that depend on the weights w_0, w_1, and w_2. Settign $\Delta = 0$, $\Gamma = 0$, and $\Lambda = 0$ is a system of three linear equations for the three unknown weights. Use the Excel function MatInv to solve this linear system and find the weights.

2. The Vasicek and Cox Ingersoll Ross models are

$$
\begin{aligned}
dr &= a\left(\bar{r} - r\right) dt + \sigma dW \quad \text{(Vas)} \\
dr &= a\left(\bar{r} - r\right) dt + \sigma\sqrt{r}dW \quad \text{(CIR)}
\end{aligned}
$$

(a) Suppose $r(0) = r_0 > 0$ is the spot price. Find a formula for $\mu(t) = E[r(t)]$. Hint: For (Vas):

$$
\begin{aligned}
d\mu &= E[dr] \\
&= E\left[a\left(\bar{r} - r\right) dt\right] \\
&= a\left(\bar{r} - \mu\right) dt
\end{aligned}
$$

so $\dot{\mu} = -a\left(\bar{r} - \mu\right)$ with $\mu(0) = r_0$.

(b) Derive a differential equation for

$$
v(t) = \text{var}[r(t)] = E\left[\left(r(t) - \mu(t)\right)^2\right]
$$

for each of the models. Find $\lim_{t\to\infty} v(t)$ in each case. Hint: solve for v in the equation $\dot{v} = 0$ with $\mu = \bar{r}$. Show that $v \to$ this as $t \to \infty$.

(c) The results of part (a) and part(b) indicate (but do not completely prove for CIR) that the probability distribution for $r(t)$ has a limit as $t \to \infty$. This is the equilibrium distribution that gives the name equilibrium model to these models. Is there such a limit for $dS = \mu S dt + \sigma S dW$?

(d) Let

$$
A(T) = \frac{1}{T} \int_0^T r(t)dt \ .
$$

Show that $A(T) \to \bar{r}$ as $t \to \infty$. Hint: For Vasicek,

$$
rdt = \bar{r}dt - \frac{1}{a}dt + \frac{\sigma}{a}dW \ ,
$$

so

$$
\frac{1}{T} \int_0^T f(t)dt = \bar{r} - \frac{1}{aT}\left(r(T) - r_0\right) + \frac{\sigma}{aT}W(T) \ .
$$

Argue that the last two terms go to zero in some sense as $T \to \infty$. Note that $W(T)$ is of order $\sqrt{T}$ generally speaking.

(e) For Vasicek only, evaluate

$$E\left[\exp\left(-\int_0^T r(t)dt\right)\right] = e^{-TY(0,T)} .$$

Find a formula for the long time effective yield

$$\overline{Y} = \lim_{T\to\infty} Y(0,T) .$$

Show that $\overline{Y} \neq \overline{r}$. The long term effective yield is not the average spot rate. Which is larger?

(f) Comment on the long term behavior of the yield curve in the Vasicek and CIR models (assuming that part (e) applies to CIR, too).

(g) *Discussion:* Though the CIR and Vasicek models can produce some realistic looking yield curves, there are other curves they cannot reproduce. If we had more time, we could use a forward Euler method to get prices for all kinds of interest rate products using these models.