Notes for Ordinary Differential Equations

Complex Exponentials in differential equations

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Complex numbers and complex exponentials simplify many calculations related to differential equations in and elegant way. Formulas from trigonometry are easy to understand by considering them to be basic properties of complex exponentials and complex multiplication. Integrals with exponentials and sines or cosines are no harder to solve than integrals involving only exponentials.

A complex number is something of the form

$$z = x + iy \ ,$$

where x and y are real numbers. The rules for adding and multiplying complex numbers are the same as the ordinary rules of algebra, if we think of i as a variable with

$$i^2 = -1 \ .$$

It is easy to check many properties of complex arithmetic:

- Commutatitivity of multiplication: $z_1 \cdot z_2 = z_2 \cdot z_1$.
- Commutatitivity of addition: $z_1 + z_2 = z_2 + z_1$.
- Associativity of addition: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.
- Associativity of multiplication: $(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$.
- Distributitivity: $z_1 \cdot (z_2 + z_3) = (z_1 \cdot z_2) + (z_1 \cdot z_3)$.
- Multiplicative inverse (division): for any $z \neq 0$ there is a unique u with $uz = 1$. We write this as $u = 1/z$.

Graphically, we think of a complex number as a point in the *complex plane*, with the horizontal axis being the *real axis* and the vertical axis being the *imaginary axis*. If $z = x + iy$, then the *real part* is $\mathrm{Re}(z) = x$, and the *imaginary part* is $\mathrm{Im}(z) = y$. The real part is the coordinate on the real axis and the imaginary part is the coordinate on the imaginary axis. The *complex conjugate* of z is $\overline{z} = x - iy = \mathrm{Re}(z) - i\mathrm{Im}(z)$. It is the image of z reflected through the real axis. You can check that if z and w are two complex numbers, then $\overline{(z + w)} = \overline{z} + \overline{w}$, and that $\overline{(z \cdot w)} = \overline{z} \cdot \overline{w}$. The *norm*, or *modulus* of z is the distance from (x, y) to the origin in the complex plane: $|z| = \sqrt{x^2 + y^2}$. You can check that $|zw| = |z|\,|w|$, and that $|z|^2 = \overline{z}z$ (complex multiplication). In fact the first property may be proven from the second:

$$\begin{aligned}
|zw|^2 &= \overline{(zw)} \cdot (zw) \\
&= (\overline{z}z) \cdot (\overline{w}w) \qquad \text{(commutatitivity of complex multiplication)} \\
&= |z|^2\,|w|^2 \ .
\end{aligned}$$

If $f(x)$ is a function of x, it is likely that we can *extend* f to $f(z)$, defined also when its argument is complex[1]. This clearly is possible when f involves only the elementary arithmetic operations, such as $f(x) = \frac{x^2+1}{x^2-2}$, which extends to $f(z) = \frac{z^2+1}{z^2-2}$. We soon will see how to define other powers, such as $\sqrt{z}$. Our goal is to define the complex exponential e^z.

We can describe the location of a complex number in the complex plane using polar coordinates, $z = x + iy$, where $x = r\cos(\theta)$ and $y = r\sin(\theta)$. The polar coordinates are $r = |z|$, and θ, the angle formed with the real axis. We call θ the *argument* of z and write $\theta = \text{Arg}(z)$. The argument is not defined uniquely since we may add or subtract 2π without moving z. All other things being equal, we generally choose the θ between 0 and π for z in the *upper half plane* ($\text{Im}(z) > 0$) and $-\pi < \theta < 0$ for z in the *lower half plane*. Others choose θ with $0 \leq \theta < 2\pi$ as the default. For example, we could write $\text{Arg}(-i) = -\frac{\pi}{2}$ or $\text{Arg}(-i) = \frac{3\pi}{2}$ (the two default conventions), but $\text{Arg}(-i) = \frac{7\pi}{2}$ also works. The geometric rule for multiplying complex numbers is: multiply the lengths, add the angles. The first part we saw already: $|zw| = |z|\,|w|$. The second we can (but refrain at the moment) verify using angle sum formulas from trigonometry. This rule is one reason not to be strict about which argument of z we choose. For example, note that $i^5 = i^2 i^2 i = (-1)\cdot(-1)\cdot i = i$. If we take the standard value, $\text{Arg}(i) = \frac{\pi}{2}$, then

$$\text{Arg}(i^5) = \text{Arg}(i \cdot i \cdot i \cdot i \cdot i) = \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} = \frac{5\pi}{2} \; .$$

Thus, following the geometric rules of complex multiplication leads to arguments outside the conventional range. We should get used to that. Should we forget that "the" argument is not uniquely defined, we might get apparent contradictions such as: $i^5 = i$, so $\text{Arg}(i^5) = \text{Arg}(i)$ and therefore $\frac{5\pi}{2} = \frac{\pi}{2}$ (which is not true).

The key to the complex exponential is the remarkable formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \; . \tag{1}$$

More generally,

$$e^z = e^{x+iy} = e^x e^{iy} = e^x\left(\cos(y) + i\sin(y)\right) = e^x\cos(y) + ie^x\sin(y) \; . \tag{2}$$

There are many ways to derive this formula and we eventually will see several of them. However, we cannot prove it because it really is the definition of the complex exponential. What we can do is to show that this definition is consistent with other reasonable definitions and that the complex exponential has many properties of ordinary exponentials.. The two important ones are

$$e^{z+w} = e^z \cdot e^w \; , \tag{3}$$

[1]The extended function is the same as the original when the argument is real, i.e. when $\text{Im}(z) = 0$.

and

$$e^w = 1 + w + O(|w|^2) \ . \tag{4}$$

For solving differential equations, the crucial property of the complex exponential is that, for any complex number z,

$$\frac{d}{dt}e^{zt} = ze^{zt} \ . \tag{5}$$

We could verify this by differentiating

$$e^{(x+iy)t} = e^{xt}\cos(yt) + ie^{xt}\sin(yt) \ ,$$

but we prefer to do it directly using the properties (3) and (4)

$$\begin{aligned}
\frac{d}{dt}e^{zt} &= \lim_{\Delta t \to 0} \frac{e^{z(t+\Delta t)} - e^{zt}}{\Delta t} \quad \text{(def. of derivative)} \\
&= \lim_{\Delta t \to 0} \frac{e^{zt}e^{z\Delta t} - e^{zt}}{\Delta t} \quad \text{(property (3))} \\
&= \lim_{\Delta t \to 0} e^{zt}\frac{e^{z\Delta t} - 1}{\Delta t} \quad \text{(distributivity)} \\
&= e^{zt}\lim_{\Delta t \to 0} \frac{e^{z\Delta t} - 1}{\Delta t} \quad \text{(pull a constant out of the limit)} \\
&= e^{zt}\lim_{\Delta t \to 0} \frac{z\Delta t + O(\Delta t^2)}{\Delta t} \quad \text{(property (4) with } w = z\Delta t) \\
&= e^{zt}z \ .
\end{aligned}$$

Complex exponentials are a great computational aid, but the eventual solution of a differential equation probably is real. One way to get a real function from a complex exponential is to take the real or imaginary part. For example, we can express simple oscillations in terms of complex exponentials as

$$\cos(\omega t) = \text{Re}(e^{\imath\omega t}) \ , \quad \text{and} \ \sin(\omega t) = \text{Im}(e^{i\omega t}) \ .$$

Conversely, we can manipulate sine or cosine functions using complex exponentials.

Example. Use the complex exponential to calculate

$$I = \int_0^t \sin(4s)ds \ .$$

Since integration is just addition and $\text{Re}(z_1 + z_2 + \cdots) = \text{Re}(z_1) + \text{Re}(z_2) + \cdots$, we have (and also for the imaginary part)

$$\int \text{Re}(f(s))ds = \text{Re}\left(\int f(s)ds\right) \ .$$

Using $\sin(4s) = \text{Im}(e^{4is})$, we find $I = \text{Im}(J)$, where

$$J = \int_0^t e^{4is}ds \ .$$

Now, from the differentiation formula (5), we can find the antiderivative:

$$e^{4is} = \frac{d}{ds}\frac{1}{4i}e^{4is} \ ,$$

so (note $\frac{1}{i} = -i$ because $i \cdot (-i) = -i^2 = -(-1) = 1$):

$$\begin{aligned}
\int_0^t e^{4is} ds &= \frac{1}{4i} \cdot e^{4is}\Big|_0^t \\
&= \frac{1}{4i}\left(e^{4it} - 1\right) \\
&= \frac{-i}{4}\left(\cos(4t) - 1\right) + \frac{-i}{4} i \sin(4t) \\
&= \frac{1}{4}\sin(4t) + i \cdot \left(\frac{-1}{4}\left(\cos(4t) - 1\right)\right) \ .
\end{aligned}$$

The imaginary part is in big parentheses on the bottom right:

$$\int_0^t \sin(4s) ds = \mathrm{Im}\left(\int_0^t e^{4is} ds\right) = \frac{1 - \cos(4t)}{4} \ .$$

Example. Solve the differential equation

$$\frac{dx}{dt} + kx = \sin \omega t \quad , \qquad x(0) = 0 \ .$$

We can write $x(t) = \mathrm{Im}(z(t))$, where

$$\frac{dz}{dt} + kz = e^{i\omega t} \quad , \qquad z(0) = 0 \ .$$

Using the integrating factor $\mu(t) = e^{kt}$, we get

$$\frac{d}{dt}\left(e^{kt} z(t)\right) = e^{kt} e^{i\omega t} = e^{(k+i\omega)t} \ .$$

Taking the indefinite integral of both sides gives

$$e^{kt} z(t) = \frac{1}{k + i\omega} e^{(k+i\omega)t} + C \ ,$$

so

$$z(t) = e^{-kt}\frac{1}{k + i\omega} e^{(k+i\omega)t} + Ce^{-kt} = \frac{1}{k + i\omega} e^{i\omega t} + Ce^{-kt} \ .$$

The initial condition $z(0) = 0$ gives

$$0 = \frac{1}{k + i\omega} + C \ ,$$

so

$$z(t) = \frac{1}{k + i\omega} e^{i\omega t} - \frac{1}{k + i\omega} e^{-kt} \ .$$

To find the answer $x = \mathrm{Im}(z)$, we have to calculate the imaginary parts of the two terms on the right. The second term involves only a real exponential, so we calculate

$$\frac{1}{k+i\omega} = \frac{1}{k+i\omega}\frac{k-i\omega}{k-i\omega} = \frac{k-i\omega}{k^2+\omega^2} \ ,$$

so

$$\mathrm{Im}\left(\frac{-1}{k+i\omega}e^{-kt}\right) = \frac{\omega e^{-kt}}{k^2+\omega^2} \ .$$

For the first term we have (note that if b is real, then $\mathrm{Im}(bz) = b\mathrm{Im}(z)$):

$$\begin{aligned}\mathrm{Im}\left(\frac{1}{k+i\omega}e^{i\omega t}\right) &= \frac{1}{k^2+\omega^2}\mathrm{Im}\big((k-i\omega)(\cos(\omega t)+i\sin(\omega t))\big) \\ &= \frac{1}{k^2+\omega^2}\big(k\sin(\omega t)-\omega\cos(\omega t)\big) \ .\end{aligned}$$

Altogether, we get

$$x(t) = \frac{1}{k^2+\omega^2}\big(k\sin(\omega t)-\omega\cos(\omega t)\big) + \frac{\omega e^{-kt}}{k^2+\omega^2} \ .$$

This is what we got before. The answer is not simpler, but the derivation is.

The complex exponential gives a simple way to remember certain trigonometric identities. For example, $e^{i\theta_1}\cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$ gives

$$\big(\cos(\theta_1)+i\sin(\theta_1)\big)\big(\cos(\theta_2)+i\sin(\theta_2)\big) = \cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2) \ .$$

Multiplying out the left side and equating real and imaginary parts gives

$$\cos(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2) = \cos(\theta_1+\theta_2) \ ,$$

(note the minus sigh which comes from $i^2 = -1$) and

$$\cos(\theta_1)\sin(\theta_2)+\cos(\theta_2)\sin(\theta_1) = \sin(\theta_1+\theta_2) \ .$$

The polar coordinate representation of a complex number is $z = re^{i\theta}$. If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then

$$z_1z_2 = r_1e^{i\theta_1}r_2e^{i\theta_2} = r_1r_2e^{i\theta_1}e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)} \ .$$

This demonstrates (but still does not prove) the rule: multiply the lengths, add the angles.

We have seen that $z(t) = e^{i\omega t}$ corresponds to simple oscillation. As time increases, $z(t)$ moves with constant speed *angular velocity* ω in a counterclockwise direction around the unit circle. The real part goes back and forth sinusoidally (or "cosinusoidally") along the real axis while the imaginary part goes up and down the imaginary axis. More generally, consider the function $z(t) = Ae^{i\omega(t-t_0)}$, where $A > 0$ is the *amplitude* and t_0 is the *phase lag* of the oscillation. This moves with steady angular velocity about a circle of radius A

and $|z(t)| = A$ always. The t_0 is a lag because, for example, $Ae^{i\omega(t-t_0)}$ "upcrosses" the real axis (goes from below to above) at time t_0 while the simpler formula $z(t) = e^{i\omega t}$ upcrosses when $t = 0$. If we multiply this out, we get

$$\mathrm{Im}(z(t)) = A\big(\cos(\omega t_0)\sin(\omega t) - \sin(\omega t_0)\cos(\omega t)\big) ,$$

which agrees with what we got (in a more complicated way) in Homework 1.

Exercises.

1. If $z = x + iy$, $w = u + iv$, and $zw = a + ib$, verify by direct calculation without complex exponentials that $a^2 + b^2 = (x^2 + y^2)(u^2 + v^2)$. What rule of complex multiplication does this verify?

2. For $f(z) = \frac{z^2+1}{z^2-2}$, evaluate $f(i)$ and $f(1 + i)$

3. Calculate $(1 + i)^2$ in two ways:

   (a) Just do it.

   (b) Find $r = |1 + i|$ and $\theta = \mathrm{Arg}(1 + i)$, then find the complex number with length r^2 and argument 2θ graphically.

4. If $w = e^z$, then $\sqrt{w} = \pm e^{z/2}$. Use this to calculate the square root of $i = e^{i\pi/2}$. Check your answer by multiplication.

5. Use complex exponentials and the binomial formula $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ to verify the formula $\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$.

6. (a) Verify that $\displaystyle\int_0^{2\pi} e^{in\theta} d\theta = 0$ whenever n is an integer and $n \neq 0$.

   (b) Verify the formula $\cos(\theta) = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$.

   (c) Use these facts to calculate $\displaystyle\int_0^{2\pi} \cos^2(\theta) d\theta$.