Ordinary Differential Equations

Homework 12

Given: December 8
Due: December 13

1. Use the formula $\sin(x) = \left(e^{i\theta} - e^{-i\theta}\right)/2i$ to derive

$$\int_0^\pi \sin^2(nx)dx = \frac{\pi}{2} \;, \tag{1}$$

and

$$\int_0^\pi \sin(nx)\sin(mx)dx = 0 \;\;,\;\; \text{if } n > 0,\, m > 0, \text{ and } n \neq m. \tag{2}$$

2. If

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx) \quad \text{for } 0 < x < \pi, \tag{3}$$

find a formula for a_n. Hint: multiply (3) by $\sin(mx)$ and integrate between 0 and π. Then use (1) and (2). This is the Fourier sine series for the function f in the interval $(0, \pi)$. The numbers a_n are the Fourier sine coefficients of f.

3. Calculate the Fourier sine coefficients for $f(x) = x$. Does the formula (3) hold for x outside the interval $(0, \pi)$? Hint: the Fourier sine series on the right side of (3) is a periodic function of x.

4. We have a rod of length π and initial temperature distribution $u_0(x) = x$ for $0 < x < \pi$. Write the solution to the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \;,$$

with Dirichlet boundary conditions $u(0, t) = u(\pi, t) = 0$ for all t, as an infinite sum of exponential solutions.

5. For large t, which is the largest term in this sum? By what factor is it larger than the next largest term when $\alpha = 1$ and $t = 2$? Identify the terms, then use a calculator to evaluate the ratio. Comment on the ratio. Suggest an approximation for $u(x, t)$ that does not involve a sum that is valid for large t.

6. For $f(x)$ given by a Fourier sine series (3), show that

$$\int_0^\pi f(x)^2 dx = Const \cdot \sum_{n=1}^{\infty} a_n^2 \;. \tag{4}$$

This is a *Parseval relation*, a sort of Pythagorean theorem for Fourier series. Find the constant. Hint: in addition to (3), also write $f(x) = \sum_m a_m \sin(mx)$ so that

$$\begin{aligned} f^2(x) &= f(x) \cdot f(x) \\ &= \left(\sum_{n=1}^{\infty} a_n \sin(nx) \right) \cdot \left(\sum_{m=1}^{\infty} a_m \sin(mx) \right) \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n a_m \sin(nx) \sin(mx) \ . \end{aligned}$$

Integrate this and use (1) and (2) to get (4) and identify the constant.

7. Apply the Parseval relation (4) to $f(x) = x$ using the Fourier coefficients you found in question 3 to get a formula for

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

in terms of π. This is yet another remarkable way π turns up.

8. Suppose a rod is made of radioactive material that produces heat at a uniform rate uniformly throughout the bar. The heat equation for this situation is

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + m \ ,$$

where the constant m is determined by the rate of heat production. After a long time $\partial u / \partial t \to 0$. Assuming $\partial u / \partial t = 0$, find the temperature distribution. Assume Dirichlet boundary conditions for a rod of length L: $u(0) = 0$, $u(L) = 0$.