Ordinary Differential Equations

Homework 2

Given: September 16
Due: September 20

Read the posted notes on complex exponentials and do the exercises there.
Complete the calculation we did in class Thursday. We had the differential equation

$$\dot{x} = -1 - x^2 ,$$

which we rewrote as

$$\frac{dx}{x^2+1} = -dt$$

then integrated to

$$\int \frac{dx}{x^2+1} = -t + C . \tag{1}$$

We solved the integral on the left using the method of partial fractions. The roots of the quadratic x^2+1 in the denominator are i and $-i$ so we have the factorization $x^2+1 = (x+i)(x-i)$ (check it). The partial fractions expansion then is

$$\frac{1}{x^2+1} = \frac{1}{(x+i)(x-i)} = \frac{A}{x+i} + \frac{B}{x-i} .$$

a. Calculate the numbers A and B, then use the formula $\int \frac{dx}{x-a} = \ln(x-a)$ (which is true for real or complex numbers a) to derive the expression (which we had in class)

$$x = -i \left(\frac{1 + Ce^{-2it}}{1 - Ce^{-2it}} \right) . \tag{2}$$

The constant C here may not be the same as the constant in (1).

b. Use the *real* initial condition $x(0) = x_0$ to find

$$C = \frac{ix_0 - 1}{ix_0 + 1} . \tag{3}$$

c. Use the amazing formula $e^{it} = \cos(t) + i\sin(t)$ to derive the formulas

$$\cos(t) = \frac{e^{it} + e^{-it}}{2} \quad , \quad \text{and} \quad \sin(t) = \frac{e^{it} - e^{-it}}{2i} . \tag{4}$$

d. Substitute (3) into (2), then multiply the numerator and denominator by $(ix_0 + 1)e^{it}$ and use (4) to get the formula

$$x(t) = \frac{x_0 \cos(t) - \sin(t)}{\text{something similar}} \; . \tag{5}$$

Of course, you are supposed to calculate the "something similar". The algebra for this step contains some big formulas. Please make them big and easy to read.

e. Check that (5) satisfies the initial condition (easy) and the differential equation (a bit of calculation).

f. In class, the differential equation was a model of something slowing down ($x(t)$ was the positive speed at time t) under both Coulomb friction (force $= -1$) and nonlinear friction (force $= -x^2$). Show that the object stops moving at a time that does not go to infinity as $x_0 \to \infty$ (Hint: (5) is zero if the numerator is, and the numerator changes sign between $t = 0$ and $t = \frac{\pi}{2}$). What does this say about the effect of nonlinear friction relative to pure Coulomb friction or Coulomb plus linear friction?

g. Note that the solution $x(t)$ is defined even after the time when $x(t) = 0$. However, this mathematical formula has no relation to physical reality because the differential equation model is different when $x < 0$. Both the Coulomb and nonlinear friction forces would be positive rather than negative.