Ordinary Differential Equations

Homework 6

Given: October 14
Due: October 19

1. We discussed in class approximate $\delta-$functions that have integral equal to one but "live" near zero (see part c below). In this sequence of steps you will solve a differential equation being forced by an approximate $\delta-$function and show that this converges to the solution forced by the actual $\delta-$function, the *impulse response*, as $\epsilon \to 0$. Aside from being good practice working with differential equations, this should build your understanding and confidence reasoning about $\delta-$functions. Recall that the *Heaviside function* is $H(t) = 1$ if $t > 0$ and $H(t) = 0$ for $t < 0$. In the $60'^s$ this might have been called an instant turn-on.

   (a) Find the solution of

   $$\ddot{x} + 4x = H(t) \quad \text{with } x(t) = 0 \text{ for } t < 0.$$

   One method, the ansatz $x(t) = a + b\cos(\omega t) + c\sin(\omega t)$ for $t > 0$ and $x(0) = 0$, $\dot{x}(0) = 0$.

   (b) Find the solution of

   $$\ddot{x} + 4x = H(t - t_0) \quad \text{with } x(t) = 0 \text{ for } t < t_0.$$

   Hint: don't re-solve the equation, just adjust the solution to part a. The right side is a *shifted* Heaviside function, shifted in time by t_0.

   (c) Express the approximate $\delta-$function

   $$f_\epsilon(t) = \begin{cases} \frac{1}{\epsilon} & \text{if } 0 \le t < \epsilon. \\ 0 & \text{otherwise.} \end{cases}$$

   as the sum (or difference) of (multiplies of possibly) shifted Heaviside functions.

   (d) Find the solution of

   $$\ddot{x}_\epsilon + 4x_\epsilon = f_\epsilon(t) \quad \text{with } x_\epsilon(t) = 0 \text{ for } t < 0.$$

   Hint: use parts a, b, c above and superposition.

   (e) Find

   $$\lim_{\epsilon \to 0} x_\epsilon(t) = Y(t)\ .$$

   (Show that the limit exists, figure out what it is, call the answer $Y(t)$.)

(f) Verify that this is the impulse response, i.e., that $Y(t)$ from part e satisfies $\ddot{Y} + 4Y = 0$ (by differentiating) for $t > 0$ and that $Y(0) = 0$, $\dot{Y}(0) = 1$.

(g) Consider the general convolution integral involving the impulse response function

$$x(t) = \int_{-\infty}^{t} Y(t-s)g(s)ds \ .$$

Compute the second derivative of the right side and show that

$$\frac{d^2}{dt^2}x(t) = -4x(t) + g(t) \ .$$

Hint: when you compute $\frac{d}{dt}\int_{-\infty}^{t} Y(t-s)g(s)ds$ there are two terms because the integral depends on t in two ways, both in the integrand and as the limit of integration. If you're uncertain how to do this, you can fall back on the chain rule from multivariate calculus. Define a function

$$F(u,v) = \int_{-\infty}^{u} Y(v-s)g(s)ds \ .$$

Note that our integral is $F(t,t)$, so (because $\frac{du}{dt} = \frac{dv}{dt} = 1$):

$$\frac{d}{dt}F(t,t) = \frac{\partial F}{\partial u} + \frac{\partial F}{\partial v} \ .$$

The first term on the right is the change coming from the change of the range of integration and the second is the term coming from the change of the integrand.

2. (Like section 3.7). For each problem, find the impulse response, express the particular solution as a convolution integral, then calculate the integral. The problems also can be solved by the ansatz method (method of undetermined coefficients). Of course, once you have the solution, the ansatz is easy to guess. Nevertheless, show that the same answer would come from the ansatz method. Don't forget to check that these answers actually satisfy the differential equation. Work oscillatory integrals using the method of complex exponentials.

(a) $\ddot{x} - 5\dot{x} + 6x = 2e^t$.

(b) $\ddot{x} + 2\dot{x} + 2x = te^t$.

(c) $\ddot{x} + 2\dot{x} + 2x = \sin(t)$ $(= \mathrm{Im}(e^{it})$.