Ordinary Differential Equations

Homework 6

Given: October 21
Due: October 26

Section 6.2: #1, 2, 4, 5, 11, 15, 20, 25.
Sample midterm

1. Find the exponential solutions of the equation $y'' + 2y' - y = 0$. For each of these, say which of the following behaviors it exhibits:
   - Simple exponential growth
   - Simple exponential decay
   - Exponential growth with oscillation
   - Exponential decay with oscillation.
2. Find particular solutions for $\ddot{x} + 4\dot{x} + 5x = f(t)$, where
   (a) $f(t) = te^{-2t}$
   (b) $f(t) = \sin(t)$
   (c) $f(t) = \sin(t)e^{-2t}$
3. (a) Find the general solution of
$$\ddot{y} + 3\dot{y} + 2y = t\ . \tag{1}$$
   (b) Use this to find the solution of (1) that has $y(0) = 1$, $\dot{y}(0) = -1$.
4. A spring – mass – damper (friction) system has spring constant[1]
$$k = 10\frac{\text{Newtons}}{\text{cm}}\ ,$$
   and a friction constant
$$\gamma = .2\frac{\text{Newtons} \cdot \text{sec}}{\text{cm}}\ .$$
   (a) Find the mass m_* (in kilograms) that makes the system critically camped.
   (b) True or false (explain your answer): If $m < m_*$, the system will have oscillatory decay, and for $m > m_*$ it will have simple exponential decay.

[1] Physicists do not like to mix Newtons, which is the force that accelerated one kilogram by one meter per second per second, with centimeters. They would work either in MKS (meter, kilogram, second) units with forces in Newtons and masses in kilograms, or in cgs (centimeter, gram, second) units with masses in grams and forces in dynes (makes one gram accelerate by one centimeter per second per second). However, normal springs have forces of the order of Newtons and displacements of the order of centimeters.

5. (a) Find the fundamental solution (impulse response) to $\ddot{y}+3\dot{y}+2y = f(t)$ (left side of (1)) by solving the initial value problem with $f = 0$ and $y(0) = 0$, and $\dot{y}(0) = 1$.

   (b) Express the solution of $\ddot{y} + 3\dot{y} + 2y = f(t)$ as an integral involving the fundamental solution and f.

   (c) Let $f(t)$ be the function $f(t) = 0$ for $t < 0$, $f(t) = t^2$ for $0 < t < 1$, and $f(t) = 0$ for $t > 1$. Make a graph of $f(t)$ for $-1 < t < 2$.

   (d) Work the integral from part (b) with the function of part (c) to find the solution of the differential equation. Note that the answer depends on whether $t < 0$, $0 < t < 1$, or $t > 1$. This is a bit time consuming, but it will be very instructive to do it carefully and completely.

   (e) Show that the formulas from part (c) have y and $\dot{y}$ continuous at $t = 0$ and $t = 1$ but that $\ddot{y}$ is not continuous at $t = 1$.

6. (a) Calculate the Laplace transform of $f(t) = te^{at}$.

   (b) Use the result of part (a) (without doing another integral) to find the Laplace transform of $f(t) = t\sin(t)e^{-2t}$ and $f(t) = t\cos(t)e^{-2t}$.

   (c) Find the solution of the initial value problem

$$\ddot{x} + 4\dot{x} + 5x = \sin(t)e^{-2t} \quad , \quad x(0) = 1 \; , \;\; \dot{x}(0) = 0 \; .$$