Ordinary Differential Equations, Fall 2005, NYU
Practice for quiz 2.

Information. The actual quiz will follow these rules. Please read them and think about your quiz strategies now.

- Put all answers in the exam book. Do not hand in the question sheet.
- Do not guess. You will receive a few points (depending on the question) for giving no answer. If your answer is completely wrong, you will get zero points. If you do not want to answer a part of the question, state which part you are not answering. For example: "The solution is ... but I don't know how to find the blowup time."
- Give only one answer. You will lose points for incorrect statements even if they are irrelevant to the question or if you also give a completely correct answer. Cross out anything you feel is wrong.

Questions

1. Copy the equation $\int_{-\omega}^{\delta} \phi(\lambda\xi)d\xi = \dfrac{\eta - \mu\psi}{\alpha+\beta+\gamma}$. Be careful to make all the Greek characters (letters) look something like they do here.

2. (a) Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} -2 & 6 \\ 6 & 7 \end{pmatrix}$. Check your answers.

   (b) Give two linearly independent exponential solutions of $\dot{x} = Ax$.

   (c) Find ψ^1 so that $\dot{\psi}^1 = A\psi^1$ and $\psi^1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Find ψ^2 so that $\dot{\psi}^2 = A\psi^2$ and $\psi^2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Hint: express each as a linear combination of exponential solutions.

   (d) Let $\Psi(t)$ be the 2×2 matrix whose first column is ψ^1 and whose second column is ψ^2. Show that $\dot{\Psi} = A\Psi$, $\Psi(0) = I$.

   (e) Let $v = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, and show that $x(t) = \Psi(t)v$ satisfies $\dot{x} = Ax$, $x(0) = v$.

   (f) Let $w(t) = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$. Calculate $y(t) = \int_0^t \Psi(t-s)w(s)ds$. Check that $y(t)$ satisfies $\dot{y}(t) = Ay(t) + w(t)$ and $y(0) = 0$.

3. Consider the third order scalar differential equation

$$\frac{d^3y}{dt^3} - 4\frac{d^2y}{dt^2} + 6\frac{dy}{dt} - 4y = 0 \ . \qquad (1)$$

(a) Find the characteristic equation and all simple exponential solutions $y(t) = e^{rt}$. There should be three different r values. Because the characteristic equation is a cubic, at least one of the roots must be real. You should be able to find it by sketching the graph. Once you have one root, you can divide by that factor and reduce to a quadratic.

(b) Formulate the equation (1) as a first order system of the form $\dot{x} = Ax$, where A is a 3×3 matrix and x is a column vector with three components.

(c) Calculate the characteristic polynomial $p(r) = \det(A - rI)$. Compare this characteristic polynomial to the one you found in part a.

4. For each of the following matrices, sketch the phase plane vector field $\dot{x} = Ax$ to decide what kind of behavior the solution has. The possibilities include: stable spiral, unstable spiral, periodic motion, combinations of simple exponential stable and/or unstable motion. After that, find the eigenvalues and eigenvectors and see how the predictions of the phase plane portraits relate to the predictions from formal stability theory.

(a) $A = \begin{pmatrix} -3 & 4 \\ -2 & 1 \end{pmatrix}$.

(b) $A = \begin{pmatrix} -3 & 5 \\ -2 & 3 \end{pmatrix}$.

(c) $A = \begin{pmatrix} 3 & 2 \\ -4 & -3 \end{pmatrix}$.