Assignment 10, due April 28, 2pm

About homework assignments

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- Solutions must be uploaded before class starts on the day assignments are due.
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- Please check the Brightspace forum corresponding to the assignment before you start working on the assignment and from time to time while you are working on it. There may be questions, comments, or (alas!) corrections that will help you.
- Please post any comments or questions or possible corrections on the Brightspace forum for the assignment.
- Please email the instructor director directly with personal matters including requests for a homework deadline extension.
- Be follow the **academic integrity policies** that apply to this class as explained on the **class web page**. In particular, do not submit solutions prepared by AI tools.

Codes and plots

All the codes and plots for this assignment are posted with the assignment on the assignments page.

The exercises

1. (*Eigenfunctions for diffusion with advection*) The eigenvalue/eigenvector problem for the advection/diffusion problem has features in common with the problem for pure diffusion but also some differences. There is an approach to this using what is called *Sturm Liouville theory* (see the textbook), but the approach here seems more "physical" (driven by intuition), and it generalizes more naturally to problems in 2D and 3D.

   (a) Suppose $u(x,t)$ evolves with diffusion with diffusion coefficient D and advection with advection velocity w (see supplementary notes). The total flux is the sum of the advective and diffusive fluxes. Write the corresponding PDE, which takes the form

   $$u_t = (???)u_{xx} + (??)u_x + \text{possibly other terms, possibly not .}$$

   Write this in the form $u_t = Au$ and write an expression for the operator A. Assume Dirichlet boundary conditions at $x = 0$ and $x = L$.

(b) Neglecting boundary conditions, show that the space of functions $\mathcal{S}$ that satisfy the differential equation $Av = rv$ is two dimensional and is spanned by two functions of the form $e^{\mu_1 x}$ and $e^{\mu_2 x}$.

(c) Show that if $v(x) = ae^{\mu_1 x} + be^{\mu_2 x}$, and if $v(0) = 0$ and $v(L) = 0$ (Dirichlet boundary conditions), then (as in class, for some non-zero integer n)

$$\mu_1 = \mu_2 + \frac{2\pi i n}{L} \ .$$

Use this to show that the eigenvalues r_n are negative real numbers and the real eigenfunctions $v_n(x)$ have the form $v_n(x) = e^{\alpha_n x}\sin(k_n x)$. Find expressions for α_n, k_n, and r_n.

(d) The advection velocity represents flow with speed $|w|$ either to the right (if $w > 0$) or to the left (if $w < 0$). We say it flows from the *upwind* direction to the *downwind* direction. Thus, $x = 0$ is the upwind end and $x = L$ is the downwind end if $w > 0$ if the advection velocity is positive. You are facing upwind if the background is into your face and downwind if it is into your back. Show that when L or w is large, the eigenfunctions $v_n(x)$ are larger near the downwind end than the upwind end.

(e) Show that when $w = 0$ the lowest eigenvalue (the one with the smallest absolute value) r_1 is proportional to L^{-1}, but that when $w \neq 0$ the lowest eigenvalue does not go to zero when $L \to \infty$.

(f) We saw in class that when $w = 0$ the eivenfunctions for distinct eigenvalues are orthogonal in the sense that

$$\int_0^L v_n(x) v_m(x)\, dx = 0 \ , \quad \text{if } \neq n \ .$$

Show that this is not true wen $w \neq 0$. *Hint.* Because of Dirichlet boundary conditions,

$$\int_0^L f(x) g_{xx}(x)\, dx = \int_0^L f_{xx}(x) g(x)\, dx \ , \quad \text{and} \quad \int_0^L f(x) g_x(x)\, dx = -\int_0^L f_x(x) g(x)\, dx \ ,$$

The eigenfunction equation $Av = rv$ gives an expression on v with first and second derivatives. The second derivative terms work but not the first derivative terms. Alternatively, you could use expressions for v_n and v_m and integrate explicitly.

(g) Let $\widetilde{v}_n$ be the n^{th} eigenfunction for the advection-diffusion problem with advection speed $-w$ (the same strength, opposite direction). Show that these are *bi-orthogonal* with the v_n in the sense that

$$\int_0^L \widetilde{v}_n(x)\, v_m(x)\, dx = 0 \ , \quad \text{if } \ m \neq n \ .$$

Hint. The integration by parts method in the hint for part (f) works here.

(h) Show that if u has a representation in terms of eigenfunctions of A then the *expansion coefficients* b_n are given by integrating with respect to $\widetilde{v}_n$. That is, if

$$u(x) = \sum_{n=1}^{\infty} b_n v_n(x) \ ,$$

then

$$b_n = C \int_0^L \widetilde{v}_n(x)\, u(x)\, dx \ .$$

2. (*A Fourier sine series example*) Suppose we solve normalized the heat equation

$$u_t = u_{xx}$$

with Dirichlet boundary conditions on the interval $[0, \pi]$. Take the initial data to be the quadratic

$$u(x, 0) = u_0(x) = x(\pi - x) \ .$$

(a) Verify the eigenfunction orthogonality relations

$$\int_0^\pi \sin(nx)\sin(mx)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{\pi}{2} & \text{if } m = n \end{cases}$$

Hint. The $m \neq n$ case can be done using integration by parts. The $m = n$ case can be done using the fact that the average of $\sin^2$ is $\frac{1}{2}$.

(b) Express $u_0(x)$ as a Fourier sine series

$$u_0(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \ .$$

Find a formula for b_n. *Hint.* Integration by parts, again!

(c) Use the eigenfunction orthogonality relations to verify the formula and find the constant in the Parseval relation

$$\sum_{n=1}^{\infty} b_n^2 = C \int_0^\pi u_0(x)^2 \, dx \ .$$

This should be true for any continuous function $u_0(x)$.

(d) Use the specific formulas for b_n in part (b) and the Parseval relation from part (c) to get an Euler style formula involving n^{-4} rather than n^{-2}. Does your formula involve all positive integers n or possibly only even or odd n?

(e) Use some of the above to write an infinite sum formula for $u(x, t)$.

(f) For finite dimensional ODE systems $\dot{x} = Ax$, the eigenvector/eigenvalue representation of $x(t)$ gives a solution defined for $t < 0$ and $t > 0$. In that sense, solving the ODE system with $x(0) = x_0$ specified need not be an "initial" value problem because the solution is defined also for t less than the initial time $t = 0$. Is this true for the solution formula of part (e)?