Differential Equations, Spring 2026
https://math.nyu.edu/∼goodman/teaching/DifferentialEquations2026/DifferentialEquations.html

Assignment 2, due February 3, 2pm

About homework assignments

- Upload one PDF file with homework solutions to the Brightspace page for the appropriate assignment.
- You may write assignments on paper then photograph or scan them. If you do that, please collect all the images into a single pdf file to upload. You may use handwriting or typing on a tablet and upload it in pdf format. You may use LaTeX, but this is not encouraged because LaTeX is less expressive than handwriting and (for me) takes longer to prepare.
- Solutions must be uploaded before class starts on the day assignments are due.
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- Please check the Brightspace forum corresponding to the assignment before you start working on the assignment and from time to time while you are working on it. There may be questions, comments, or (alas!) corrections that will help you.
- Please post any comments or questions or possible corrections on the Brightspace forum for the assignment.
- Please email the instructor director directly with personal matters including requests for a homework deadline extension.
- Be follow the **academic integrity policies** that apply to this class as explained on the **class web page**. In particular, do not submit solutions prepared by AI tools.

The assignment

1. Consider the pair of differential equations

$$\begin{aligned} \dot{x} &= gx \\ \dot{y} &= ty \; . \end{aligned}$$

   The x equation represents simple exponential growth with fixed constant growth rate g. You can think of the y equation as a model for "variable" (or "time dependent") growth rate $g(t) = t$. The variable growth rate in the variable growth rate equation goes to infinity as $t \to \infty$, so it is natural to expect that y solutions eventually grow faster and become larger than x solutions.

   (a) Use the separation of variables method to find the solution of the y initial value problem with initial value $y(0) = y_0$.

   (b) Use the method of integrating factors to find the solution of the y initial value problem with initial value $y(0) = y_0$. *Hint*. These solutions should agree.

   (c) Use the solution formulas for the x and y equations to show that if $x_0 = x(0) > 0$ and $y_0 > 0$, then there is a $T > 0$ so that

$$y(t) > x(t) \;\; , \quad \text{if } t > T \; .$$

2. Find the solution to the initial value problem

$$\dot{x} = e^x \quad , \qquad x(0) = 0 \; .$$

Determine the blow-up time, if there is one.

3. (*normalization/non-dimensionalization*) Differential equations often involve "physical" parameters. *Non-dimensionalization* is the process of re-formulating the differential equation to give the parameters specific, "non-dimensional" values. The process is called *scaling* or *re-scaling*, which means changing "scales".

   (a) (*re-scaling time*). Suppose the ODE is

   $$\frac{d}{dt}x(t) = f(x) \; . \tag{1}$$ ODE

   You can define T to be a new "time variable" proportional to t using

   $$T = \frac{1}{\tau}\, t \; .$$

   The scaling parameter τ defines a "time scale". Write $\widetilde{x}$ for the solution expressed as a function of the scaled time variable. This may be expressed in several ways, each of which us useful for calculations

   $$\widetilde{x}(T) = x(t) \quad , \qquad \widetilde{x}(T) = x(\tau T) \quad , \qquad x(t) = \widetilde{x}(t/\tau) \; .$$

   Show that the differential equation ([ODE]**??**) is equivalent to

   $$\frac{d}{dT}\widetilde{x} = \tau f(\widetilde{x}) \; .$$

   (b) Consider the "logistics equation"

   $$\dot{x} = ax - bx^2 \; . \tag{2}$$ L

   Show that it is possible to rescale the time variable to set small x growth coefficient a equal to 1. This means that the differential equation is equivalent to an equation of the form

   $$\frac{d}{dT}\widetilde{x} = \widetilde{x} - \widetilde{b}\widetilde{x}^2 \; .$$

   It is common to "forget" the rescaling and just say that it is possible to rescale time to set $a = 1$ in the logistics equation ([L]**??**). You might also say: "Without loss of generality, we may assume $a = 1$."

   (c) Rescaling the x variable means replacing x with the scaled variable

   $$\widetilde{x} = \frac{1}{\overline{x}}\, x \; .$$

   Show that it is possible to rescale x to put

   $$\dot{x} = x - bx^2$$

   into the form

   $$\dot{\widetilde{x}} = \widetilde{x} - \widetilde{x}^2 \; .$$

   (d) Conclude that any solution of any logistics equation (with $a > 0$ and $b > 0$) is equivalent to a normalized logistics equation with small x growth rate equal to 1 and stable fixed point at $x_* = 1$.

(e) Consider the ODE with a cubic nonlinearity

$$\dot{x} = ax - bx^2 + cx^3 \ .$$

Suppose $x = 0$ is a strictly unstable fixed point and that $b > 0$. The cubic coefficient c may have either sign. Show that it is possible to rescale t and x to put the equation into the *canonical form*

$$\dot{x} = x - x^2 + \widetilde{c}\,x^3 \ .$$

Show that the normalized cubic coefficient $\widetilde{c}$ may take any value. Is the sign of $\widetilde{c}$ the same as the sign of c?

4. A fixed point, x_*, is *strongly stable* if $f'(x_*) < 0$ (strict inequality). Suppose the ODE $\dot{x} = f(x)$ has fixed points x_1, x_2, etc. (some number of fixed points). Is it possible that two consecutive fixed points are both strongly stable? Either give an example with two neighboring fixed points both strongly stable or explain why it is not possible.

5. Consider the initial value problem

$$\dot{x} = -ax + b\cos(\omega t) \ , \quad x(0) = 0 \ .$$

(a) Show that it is possible to normalize (as in Exercize 3) to set $a = 1$ and $b = 1$, or $\omega = 1$ and $b = 1$, but it is not possible to set all three parameters equal to 1.

(b) Show that there is a *steady oscillating* solution of the differential equation, but not the initial value problem, of the form

$$x_s(t) = c\cos(\omega(t - t_0)) \ . \tag{3}$$

a

This solution oscillates at the same frequency as the forcing in the ODE but has an unknown *amplitude*, c, and *phase lag*, t_0. *Hints.* The algebra simplifies if you normalize to take $a = b = 1$. Then ω is the only parameter in the equation. There are several ways to do this. One way is to substitute the *ansatz* (a??) into the ODE and solve for the parameters c and t_0 in terms of the ODE parameter ω.

(c) (*This is simple in principle but the algebra is complicated. It's only worth one point. Save this for the end and do it only if you have time.*) Show that the solution to the initial value problem converges to the steady oscillating solution as $t \to \infty$. That is, show that

$$x(t) - x_s(t) \to 0 \ , \quad \text{as } t \to \infty \ .$$

Hint. There is an integral formula for the solution. You can do the integral.

(d) What is the limiting amplitude and phase lag as $\omega \to 0$?

(e) What is the limiting amplitude and phase as and $\omega \to \infty$?

(f) (*Don't spend a lot of time with this or worry if you're not confident of your answer. It's only worth one point. It's not even clear what the question is.*) Give "physical" or intuitive explanations for your answers to parts (d) and (e).

6. We gave seen that solutions of the linear growth differential equation $\dot{x} = ax$ grow to infinity as $t \to \infty$ but do not blow up in finite time. By contrast, we saw that any positive solution of the quadratic growth rate equation $\dot{x} = ax^2$ does blow up in finite time.

(a) Consider the *power law* growth differential equation

$$\dot{x} = x^p \ , \quad p > 0 \ .$$

Show that any positive solution blows up in finite time if $p > 1$ and goes to infinity without blowing up in finite time if $0 < p \leq 1$. *Hint.* The method of separation of variables works for any $p > 0$. *Comment.* The initial value problem with $p < 1$ and $x_0 = 0$ is problematic, as we will discuss in a future class. Always assume $x_0 > 0$ to avoid this problem.

(b) Consider a differential equation with super-linear growth but slower than any super-linear power law:

$$\dot{x} = x\log(x)\ . \qquad (4)$$ lnl

(Here, $\log(\cdot)$ is the "natural log": $\log(x) = \log_e(x) = \ln x$.) Show that the growth is super-linear but sub-power by showing that for any positive C and $p > 1$ there is an $x_* > 1$ so that $Cx < x\log(x)$ (*super-linear*) and $x\log(x) < Cx^p$ (*sub-power*) if $x > x_*$.

(c) Show that the solution of the initial value problem for (lnl ??) with initial data $x_0 > 1$ has super-exponential growth but does not blow up in finite time. Super-exponential means that for any $x_0 > 1$ and $C > 0$ and $g > 0$ there is a $T > 0$ so that

$$x(t) > Ce^{gt}\ , \quad \text{if } t > T\ .$$

Hint. Use the method of separation of variables together with the fact that

$$\frac{d}{dx}\log(\log(x)) = \frac{1}{x\log(x)}\ .$$

7. In each case, explain whether $f(x)$ is locally Lipschitz, globally Lipschitz, or neither. Drawing a sketch of the graph will probably help the explanations.

$$(a)\quad f(x) = \sin(x)$$

$$(b)\quad f(x) = \sin(x^2)$$

$$(c)\quad f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0 \end{cases}$$

$$(d)\quad f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2x & \text{if } x \geq 0 \end{cases}$$

$$(e)\quad f(x) = \sqrt{|x|}$$

Special request. Just before you upload, please estimate the time you spent on this class this week (in hours) in the following activities

- Reviewing class notes
- Reading the textbook
- Reading supplementary notes
- Solving and writing up exercises
- Finally, if you found helpful (to you) online materials, please include a link or a URL.