Assignment 3, due February 10, 2pm

About homework assignments

- Upload one PDF file with homework solutions to the Brightspace page for the appropriate assignment.
- You may write assignments on paper then photograph or scan them. If you do that, please collect all the images into a single pdf file to upload. You may use handwriting or typing on a tablet and upload it in pdf format. You may use LaTeX, but this is not encouraged because LaTeX is less expressive than handwriting and (for me) takes longer to prepare.
- Solutions must be uploaded before class starts on the day assignments are due.
- Solutions must be uploaded before class starts on the day assignments are due.
- Please check the Brightspace forum corresponding to the assignment before you start working on the assignment and from time to time while you are working on it. There may be questions, comments, or (alas!) corrections that will help you.
- Please post any comments or questions or possible corrections on the Brightspace forum for the assignment.
- Please email the instructor director directly with personal matters including requests for a homework deadline extension.
- Be follow the **academic integrity policies** that apply to this class as explained on the **class web page**. In particular, do not submit solutions prepared by AI tools.

The assignment

1. (*Practice in two variable integration to get the integral over a rectangle*) Consider the rectangle in the xy plane with corners at (a, b), (c, b), (c, d), and (a, d). Assume that $a < c$ and $b < d$, so these four point move around the rectangle in the counter-clockwise direction, starting at the bottom left. Let R be the enclosed rectangle

$$R = \{\, (x, y) \mid a \le x \le c \quad \text{and} \quad b \le y \le d \,\} \; .$$

Consider the function

$$f(x, y) = 2x + 3x^2 y + y^2 \; .$$

   (a) Find a formula for

$$u(y) = \int_{x=a}^{x=c} f(x, y)\, dx \; .$$

   (b) Find a formula for

$$v(x) = \int_{y=b}^{y=d} f(x, y)\, dy \; .$$

   (c) Use your answer from part (a) to compute the integral

$$\int_{y=b}^{y=d} u(y)\, dy \; .$$

(d) Use your answer from part (b) to compute the integral

$$\int_{x=a}^{x=c} v(x)\, dx \; .$$

(e) Explain how this illustrates the second equality

$$\iint_R f(x,y)\, dxdy = \int_{y=b}^{y=d} \left(\int_{x=a}^{x=c} f(x,y)\, dx \right) dy = \int_{x=a}^{x=c} \left(\int_{y=b}^{y=d} f(x,y)\, dy \right) dx$$

The last two multiple integrals are usually written without parentheses as

$$= \int_{y=b}^{y=d} \int_{x=a}^{x=c} f(x,y)\, dxdy = \int_{x=a}^{x=c} \int_{y=b}^{y=d} f(x,y)\, dydx$$

2. (*Implicitly defined functions*) When a planet orbits a star in an elliptical orbit (as discovered by Kepler and explained by Newton), part of the description of the orbit involves *Kepler's equation*

$$y + a\sin(y) = x \; .$$

For computing orbits, $0 \leq a < 1$, with $a = 0$ corresponding to a circular orbit and a close to 1 corresponding to a highly elliptical orbit, like a comet around our sun.

(a) Draw the graph of x as a function of y and use it to show that Kepler's equation implicitly defines y as a function of x, for a in the range given.

(b) Draw a graph to show that y is not a well defined function of x if $a > 1$. There is not a unique y value defined by a given x value.

(c) Show that if $0 \leq a < 1$, then the implicitly defined function $y(x)$ satisfies

$$\frac{1}{1+a} \leq \frac{dy}{dx} \leq \frac{1}{1-a} \; .$$

Notice that the *upper bound* (the second inequality) "goes away" when a approaches 1.

3. (*Energy conservation and dissipation*) We will call a function $V(x)$ a *potential energy* function and consider the ODE

$$m\ddot{x} = -V'(x) - \gamma\dot{x} \; . \tag{1}$$

We always think of the mass, m and the friction coefficient, γ, as positive numbers. The conclusions require this but the algebra doesn't always. The *kinetic energy* is

$$\text{KE} = \frac{m}{2}\left(\dot{x}\right)^2 \; .$$

The *total energy* is the sum of the potential and kinetic energies

$$E(t) = V(x(t)) + \frac{m}{2}\left(\dot{x}(t)\right)^2 \; .$$

(a) Show that if $V(x) = \frac{1}{2}kx^2$, then the ODE (1) becomes the linear damped harmonic oscillator ODE from class

$$\ddot{x} = -\frac{k}{m}x - \frac{\gamma}{m}\dot{x} \; .$$

(b) Derive a formula for $\frac{d}{dt}E(t)$ assuming that x satisfies (1). Use this to show that the total energy is a decreasing function of t if $\gamma > 0$ and that total energy is *conserved* (doesn't change as t changes) if there is no friction.

4. (*Expressing solutions of systems of differential equations using the matrix exponential*) Suppose $M(t)$ is given as an infinite sum

$$M(t) = \sum_{n=0}^{\infty} R_n(t) \ .$$

Differentiating *term by term* is the formula

$$\dot{M} = \sum_{n=0}^{\infty} \dot{R}_n \ .$$

[If the entries of M depend on t, then the derivative is the matrix of derivatives. For example

$$\frac{d}{dt}\begin{pmatrix} 2 & t \\ e^t & t^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ e^t & 2t \end{pmatrix} \ .$$

You should make up time dependent matrices $M(t)$ and $N(t)$ and verify by calculating the matrix products and derivatives that $\frac{d}{dt}(MN) = \dot{M}N + M\dot{N}$.] If this differentiation formula is true, we say "you can take the derivative inside the sum". Term by term differentiation is correct for finite sums and most of the time for infinite sums too. There is more on that in a mathematical analysis class.

The *matrix exponential* is (two ways of expressing the same thing)

$$\exp(M) = e^M = I + M + \frac{1}{2}M^2 + \cdots + \frac{1}{n!}M^n + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}M^n \ . \tag{2}$$

Standard mathematical convention is that the $n = 0$ term has $0! = 1$. Also $M^0 = I$ for any matrix M, so that $M^j M^k = M^{j+k}$ even if $j = 0$ or $k = 0$.

(a) Use term by term differentiation (and some tricks) to show that

$$\frac{d}{dt}e^{tA} = Ae^{tA} = e^{tA}A \ .$$

Warning. Most pairs of matrices don't commute, but e^{tA} commutes with A.

(b) Consider the first order linear autonomous (coefficients independent of t) and homogeneous (no forcing term on the right hand side) system of differential equations

$$\dot{x} = Ax \ , \quad x(0) = x_0 \ .$$

Here, A is a $d \times d$ matrix, $x(t)$ is a d component column vector, and x_0 is the d component initial data. Show that the solution is

$$x(t) = e^{tA}x_0 \ .$$

Show that $x(t)$ satisfies the ODE and the initial condition. For the latter, you can show that $e^0 = I$, where 0 is the $d \times d$ matrix with all zeros. *Comment.* The matrix exponential e^{tA} is called the *fundamental solution* because the solution to the ODE with any initial condition can be expressed in terms of it, and because it satisfies the ODE in the matrix sense (part (a)).

(c) Show that if $Av = \lambda v$ (the eigenvalue λ and the eigenvector v need not be real) then

$$e^{tA}v = e^{t\lambda}v \ .$$

Hint. Start by applying the power series (2) to v.

(d) For ordinary numbers a and b, the ordinary exponential satisfies $e^{ta}e^{tb} = e^{t(a+b)}$. Show that the matrix version of this is false:

$$e^{tA}e^{tB} \neq e^{t(A+B)} \quad , \quad \text{for all small } t \ ,$$

unless A and B commute. *Hint.* Use the Taylor series (2) only up to the order t^2 terms for the right and left sides, multiply out the left sides out to the t^2 terms and compare the results. For example, $e^{tA} = I + tA + \frac{1}{2}t^2A^2 + O(t^3)$, where $O(t^3)$ means terms with power t^3 or higher. For ordinary numbers,

$$(a + tb + t^2c + O(t^3))(d + te + t^2f + O(t^3)) = ad + t(ae + bd) + t^2(af + be + cd) + O(t^3) \ .$$

See whether you think there is a matrix version of this.

5. (*Finding real solutions to real differential equations using complex numbers*) Suppose $z(t)$ is a complex number function of t that satisfies

$$\ddot{z} + a\dot{z} + bz = 0 \ ,$$

with a and b being real numbers. Suppose $x(t) = \text{Re}(z(t))$ and $y(t) = \text{Im}(z(t))$. Show that

$$\ddot{x} + a\dot{x} + bx = 0 \quad , \quad \ddot{y} + a\dot{y} + by = 0 \ . \qquad (3)$$

In plain words, the real and imaginary parts of a complex solution are real solutions. Show that if $r^2 + ar + b = 0$, with $r = \lambda + i\mu$ (λ and μ being real), then

$$x(t) = e^{\lambda t}\cos(\mu t) \quad , \quad y(t) = e^{\lambda t}\sin(\mu t)$$

are real solutions of (3).

Special request. Just before you upload, please estimate the time you spent on this class this week (in hours) in the following activities

- Reviewing class notes
- Reading the textbook
- Reading supplementary notes
- Solving and writing up exercises
- Finally, if you found helpful (to you) online materials, please include a link or a URL.