Assignment 4, due February 19, 2pm

About homework assignments

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- Solutions must be uploaded before class starts on the day assignments are due.
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- Be follow the **academic integrity policies** that apply to this class as explained on the **class web page**. In particular, do not submit solutions prepared by AI tools.

The assignment, corrected Feb. 13, to fix exerciss 2a and 3e.

1. (*Functional relations defined by differential relations*) Suppose $x(t)$ and $y(t)$ are differentiable functions of t and the differentials satisfy the differential relation

$$(4x + y)\, dx + (x + 2y)\, dy = 0\ . \tag{1}$$

   Suppose that $x(0) = 1$ and $y(0) = 1$.

   (a) Find a formula of the form $f(x, y) = 0$ that holds for all $(x(t), y(t))$.
   (b) If $\dot{x}(0) = 2$, find $\dot{y}(0)$.
   (c) What are the possible values of $y(t)$ if you know $x(t) = 1$? Is the $t = 0$ value $y(0) = 1$ the only possibility?
   (d) What is the largest value of $x(t)$? *Hint.* First find a relation between x and y that makes $dx = 0$ in (1).

2. (*Practice with overdamped spring mass ODE*) Consider the spring mass friction system

$$m\ddot{x} = -\gamma\dot{x} - kx\ . \tag{2}$$

   Take parameters $m = 1$, $\gamma = 3$, and $k = 2$.

(a) Find the two simple exponential solutions $x_1(t) = e^{r_1 t}$ and $x_2(t) = e^{r_2 t}$.

(b) Find the a formula for $x(t)$ under the initial conditions $x(0) = 0$, and $\dot{x}(0) = 1$. Express that solution in the form $x(t) = c_1 x_1(t) + c_2 x_2(t)$. Find c_1 and c_2.

(c) Suppose $x(0) = 1$. Find $\dot{x}(0)$ so that $x(t)$ has the faster decay rate as $t \to \infty$.

3. (*Another two dimensional vector space of exponential solutions*) The *Fibonacci numbers* are a sequence x_n starting with $x_0 = 1$, $x_1 = 1$. The Fibonacci numbers, x_n, for $n = 2, 3, \cdots$ are determined by the Fibonacci *recurrence relation*

$$x_{n+1} = x_n + x_{n-1} \ . \qquad (3)$$

The first few are

$$\begin{aligned} x_2 &= x_1 + x_0 = 1 + 1 = 2 \\ x_3 &= x_2 + x_1 = 2 + 1 = 3 \\ x_4 &= x_3 + x_2 = 3 + 2 = 5 \\ x_5 &= x_4 + x_3 = 5 + 3 = 8 \\ x_6 &= x_5 + x_4 = 8 + 5 = 13 \end{aligned}$$

The ratio of successive Fibonacci numbers is

$$r_n = \frac{x_n}{x_{n-1}} \ .$$

Here are some values (thanks to a computer)

n	x_n	r_n
2	2	2
3	3	1.5
4	5	$1.667\cdots$
5	8	1.6
6	13	1.625
7	21	$1.6154\cdots$
10	89	$1.6182\cdots$
15	987	$1.618033\cdots$
20	10946	$1.618040\cdots$

You can see that the x_n grow rapidly, with x_{20} more then ten thoudsand. Also, the ratios quickly approach a limiting value. This limiting value is called the *golden mean*, or *golden ratio*. It's value is $\gamma = \frac{1}{2} + \frac{1}{2}\sqrt{5} = 1.61803398\cdots$.

We use the notation

$$x = (x_0, x_1, x_2, \cdots, x_n, \cdots)$$

for an infinite sequence of numbers starting with x_0. Such a sequence is a *fibonacci sequence* if the x_n satisfy the fibonacci recurrence relation for $n = 2, 3, \cdots$. The Fibonacci numbers are a fibonacci sequence, but you get different fibonacci sequences if you change the values of x_0 and x_1. For example, if $x_0 = 1$ and $x_1 = -1$ then $x_2 = 0$, $x_3 = -1$, etc. The corresponding fibonacci sequence is $x = (1, -1, 0, -1, -1, -2, \cdots)$. The zero sequence is $0 = (0, 0, \cdots)$. It is a "trivial" fibonacci sequence, but the numbers $x_n = 0$ do satisfy the recurrence relation.

(a) Let $\mathcal{S}$ be the set of all fibonacci sequences. Show $\mathcal{S}$ is a vector space.

(b) A sequence is a *geometric* sequence if it has the form $x_n = r^n$. Find the *characteristic polynomial* that r must satisfy for the geometric sequence $x_n = r^n$ to be in $\mathcal{S}$. Show that there are two such sequences with r values $r_+ = \gamma$ (this is the golden mean) and $r_2 = -\gamma^{-1}$.

(c) Show that if $x \in \mathcal{S}$ and $y \in \mathcal{S}$, and if $x_0 = y_0$ and $x_1 = y_1$, then $x = y$. *Comment.* Specifying x_0 and x_1 then using the recurrence relation to determine the rest of the sequence is the recurrence relation equivalent of solving the initial value problem for an ODE.

(d) Let $v \in \mathcal{S}$ and $w \in \mathcal{S}$ be the two distinct geometric Fibonacci sequences. Show that they are linearly independent and that they *span* $\mathcal{S}$. *Hint.* Feel free to review the concepts of basis, linear independence, and span from your linear algebra class. Linear independence means that if $0 = cv + dw$ (c and d are numbers and 0 is the zero sequence) then $c = 0$ and $d = 0$. For this it is enough to show that $cv_0 + dw_0 = 0$ and $cv_1 = dw_1 = 0$ implies that $c = d = 0$. Why? To show that v and w span $\mathcal{S}$ (the space of all fibonacci sequences), it is enough, for any $x \in \mathcal{S}$ to find c and d so that $x_0 = cv_0 + dw_0$ and $x_1 = cv_1 + dw_1$. Why?

(e) Find a formula for the Fibonacci numbers, $x = (1, 1, 2, 3, \cdots)$, of the form

$$x_n = c\gamma^n + d\,(-\gamma)^{-n} \ .$$

(f) Show that x_n (the Fibonacci number) is "exponentially" close to $c\gamma^n$ for large n. *Hint.* Part of this exercise is giving a mathematical meaning to "exponentially close" that is more precise than just "very very very close".

(g) Show that $r_n \to \gamma$ as $n \to \infty$ for the Fibonacci numbers.

4. (*Unstable "active" oscillators*) Consider the spring/friction/mass system (2). The *trajectory* is the curve that the points $x(t), \dot{x}(t))$ make in the "x, $\dot{x}$ plane".

(a) Let the parameters be $m = 2$, $\gamma = 0$ (no friction), and $k = 4$. Show that the trajectory is an ellipse. That is, the solution $(x(t), \dot{x}(t))$ "traces out" the same ellipse over and over as t increases.

(b) Show that if $\gamma < 0$ but γ is close to zero with $m = 2$ and $k = 4$, then the trajectory is an outward moving elliptical spiral. More technically, solutions have $x(t) = e^{\lambda t}\cos(\omega(t - t_0))$, with λ slightly bigger than zero, and a related expression for $\dot{x}(t)$. If λt (in the exponent) were fixed, it would be an exact ellipse, but with λt slowly increasing with t, the ellipse slowly spirals outward. *Comment.* A passive spring/mass system cannot have negative friction, but a mass with a motor or some external force could act like a system with negative friction. We will see examples of that later in the class.

5. (*ODE problems as linear algebra*) Suppose $x(t)$ satisfies a second order linear ODE with unknown coefficients but whose solutions can be measured in experiments. The experiment measures solutions $x_1(t)$ and $x_2(t)$. The experiment is set up so that the initial condition is set and the corresponding solution and its derivative at time T are measured. Suppose initial conditions $x_1(0) = 1$ and $\dot{x}_1(0) = 0$ gives solution $x_1(T) = 2$ and $\dot{x}_1(T) = -1$ and initial condition $x_2(0) = 0$ and $\dot{x}_2(0) = 1$ gives solution $x_2(T) = -1$ and $\dot{x}_2(T) = 1$. Find the initial conditions that give $x(T) = 1$ and $\dot{x}(T) = 0$. *Method.* Because the ODE is linear and the solution space is two dimensional and spanned by x_1 and x_2, you can write $x = c_1x_1 + c_2x_2$. This allows you to find $x(T)$ and $\dot{x}(T)$ in terms of the measured data and the coefficients c_1 and c_2. Formulate this in matrix/vector form

$$Sc = \begin{pmatrix} x(T) \\ \dot{x}(T) \end{pmatrix} \ .$$

Here c is the vector of unknown coefficients to be determined

$$c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \ .$$

You need to identify the 4 entries of the 2×2 matrix S so that (We write $*$ to represent a *matrix element* that needs to be identified.)

$$Sc = \begin{pmatrix} * & * \\ * & * \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x(T) \\ \dot{x}(T) \end{pmatrix}$$

The last step is to solve the system, either by Gaussian elimination or by finding S^{-1} (or another method?) to find (explain why the vector on the right is correct)

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = S^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

6. (*Real and complex forced oscillator solutions*) Consider the "steady" solution of the "sinusoidally forced" oscillator, either in exponential form or in real form

$$\ddot{z} = -\frac{k}{m} z - \frac{\gamma}{m} \dot{z} + e^{i\omega t}$$
$$\ddot{x} = -\frac{k}{m} x - \frac{\gamma}{m} \dot{x} + \cos(\omega t) .$$

The corresponding solutions are written (we assume $A_x > 0$)

$$z(t) = A_z(\omega)\, e^{i\omega t} \quad , \quad x(t) = A_x(\omega)\, \cos(\omega(t - t_x(\omega))) .$$

The phase and amplitude of the response are functions of the driving frequency.

(a) Show that $A_x(\omega) = |A_z(\omega)|$.

(b) Use the answer to part (a) to write a formula for the *frequency response* function $A_x(\omega)$.

(c) Fix $m = 1$ and $k = 1$ (which can be done by normalization). Describe the frequency response function, when γ is small, in as much or little detail as you have time for. Simple approximate answers are preferable to complicated exact formulas. For example, $A_{\max} \approx \frac{1}{\gamma}$ (if that's right) is preferable to $A_{\max} = \sqrt{1 + \gamma + \frac{1}{\gamma^2}}$ (if that's right). Even "$A_{\max}$ is large when γ is small" is preferable to a big exact formula. Of course, an exact complicated formula can be a way to find a simpler more useful approximation. Your answer may address some or all of the following questions or related questions:

- What does the frequency response curve look like for small γ (make a sketch)?
- Where is its peak?
- How high is the peak, $A_{\max}$?
- What is the frequency of the maximum response, $\omega_{\max}$?
- What is the half-width of the response peak? This is $\Delta\omega$ so that

$$A_x(\omega_{\max} \pm \Delta\omega) = \frac{1}{2} A_{\max} .$$

(The full width is $2\Delta\omega$, which is $\omega_{\max} + \Delta\omega - (\omega_{\max} - \Delta\omega)$.) This $\Delta\omega$ is how much you can "de-tune" the forcing frequency and still get a large response.

Special request. Just before you upload, please estimate the time you spent on this class this week (in hours) in the following activities

- Reviewing class notes
- Reading the textbook
- Reading supplementary notes
- Solving and writing up exercises
- Finally, if you found helpful (to you) online materials, please include a link or a URL.