Assignment 5, due March 3, 2pm

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1. (*Normal modes of oscillators*) Figure 7.1.1 on page 288 and Exercise 14 on page 292 of the textbook explain the second order ODE system that models two masses coupled to each other and to walls by linear springs and without friction. This exercise explores the behavior of these models using eigenvalues and eigenvectors. The vector space of solutions, $\mathcal{S}$ is spanned by *modal solutions* given by complex exponentials or in sine/cosine form (2). The general ODE system $\dot{x} = Ax$ has modal solutions of the form $x(t) = ve^{rt}$ where r could be any complex number. That is because A could be any square matrix. The matrix used for the coupled oscillator problem here has "special structure", it is symmetric and positive definite. This means that its eigenvalues are positive and real and that its eigenvectors are real and orthogonal. This exercise explains some of these facts and illustrates their importance. Some of the exercise involves a general $n \times n$ symmetric positive definite matrix, K. Other parts are only about the specific K relevant for the two mass system of Figure 7.1.1.

   (a) Show that, if $m_1 = m_2 = 1$, the ODE system of Figure 7.1.1 can be expressed in the form

   $$\ddot{x} = -Kx \ , \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ , \quad K \text{ is a } 2\times 2 \text{ symmetric matrix} \ . \tag{1}$$

   Find an expression for K in terms of the spring constants k_1, k_2, and k_3.

(b) Consider an ODE system of the form (1) but with $x \in \mathbb{R}^n$ and K being a symmetric $n \times n$ matrix with positive eigenvalues. Show that if $Kv = \omega^2 v$, then (1) has complex solutions of the form $x(t) = v\, e^{\pm i\omega t}$ and real solutions of the form $x(t) = v\cos(\omega t)$ and $x(t) = v\sin(\omega t)$. This notation $e^{\pm\cdots}$ means that there are two solutions

$$x_+(t) = v\, e^{+i\omega t} \quad , \quad x_-(t) = v\, e^{-i\omega t} \; .$$

Terminology. The eigenvector is a *vibrational mode* or *oscillation mode* or just *mode*. The corresponding ω is the vibration *frequency*.

(c) Find the two oscillation frequencies, ω_1 and ω_2, and corresponding modes, v_1 and v_2, if $k_1 = k_2 = k_3 = 1$. The frequencies are unique if you take them to be positive. The modes are not unique because if v is an eigenvector then cv is also an eigenvector for any constant c.

(d) Find a formula for the solution of the initial value problem with initial conditions[1]

$$x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \qquad \dot{x}(0) = 0 \; .$$

Hint The vector space $\mathcal{S}$ of solutions is spanned by the four solutions

$$v_1\cos(\omega_1 t) \; , \;\; v_1\sin(\omega_1 t) \; , \;\; v_2\cos(\omega_2 t) \; , \;\; v_2\sin(\omega_2 t) \; . \tag{2}$$

Therefore the solution we seek can be written as a linear combination of these. You can write this linear combination as

$$x(t) = a_1\cos(\omega_1 t)\, v_1 + b_1\sin(\omega_1 t)\, v_1 + a_2\cos(\omega_2 t)\, v_2 + b_2\sin(\omega_2 t)\, v_2 \; .$$

Find the four coefficients a_1, b_1, a_2, b_2 by linear algebra using the four initial conditions (two vectors with two components each). It's easiest to do this using vectors (rather than individual numbers) as far as possible. The initial condition $\dot{x}(0) = 0$ eliminates two of the coefficients right away.

(e) Show that if all the spring constants are positive then the eigenvalues of K are both positive. *Hint.* The eigenvalue is ω^2, which satisfies a quadratic equation with coefficients involving k_1, k_2, and k_3. Once you have ω^2, you can use $\omega = \pm\omega^2$, so find a formula for ω^2 first and get ω second. The quadratic formula involves a square root. To show ω^2 is real, notice that $k_1^2 + k_3^2 - 2k_1k_3 = (k_1 - k_3)^2 \geq 0$. Here are possible approaches to showing $\omega^2 > 0$. Choose one or invent another approach.

- Clever tricks with inequalities (I'm terrible at this).
- Write $\omega^2 = a \pm \sqrt{b}$, find expressions for a and b in terms of k_1, k_2, and k_3 and then use the slow and boring method of Lagrange multipliers to find the maximum of b with a fixed. Then you can see whether $a - \sqrt{b_{\max}}$ is negative.
- The *potential energy* is

$$\text{PE} = \frac{1}{2}\left(k_1 x_1^2 + k_2(x_1 - x_2)^2 + k_3 x_2^2 \right) \; .$$

Show that $\text{PE} = \frac{1}{2}x^T K x$ by direct calculation. Since $\text{PE} > 0$ if $x \neq 0$ (assume that at least one spring constant is positive), this implies that $x^T K x > 0$ for any non-zero x. A matrix like that is called *positive definite*. Show that a positive definite symmetric matrix has positive eigenvalues by showing that if $Kv = \lambda v$, then $v^T K v = \lambda v^T v > 0$.

[1] The vector of all zeros is denoted by 0 no matter what dimension it has.

(f) Show that modes corresponding to different frequencies are orthogonal to each other. That is, if $\omega_1^2 \neq \omega_2^2$ then $v_1^T v_2 = 0$. *Hint.* First, $v_2^T v_1 = v_1^T v_2$. The eigenvalue formula $\omega^2 v = Kv$ implies that $\omega_1^2 v_2^T v_1 = v_2^T K v_1$, and that $\omega_2^2 v_1^T v_2 = v_1^T K v_2$.

(g) Use part (f) to show that the two mass coupled oscillator has one mode in which the masses always move in the opposite direction and one mode where the masses always move in the same direction. "Always move in the opposite direction" means that if $\dot{x}_1(t) > 0$ then $\dot{x}_2(t) < 0$ and conversely. Here $\dot{x}_1(t)$ is the velocity of the first mass, not the time derivative of the vector $x(t)$ in the first mode. A "modal solution" is any of the four solutions (2).

2. (*Weakly coupled oscillators, beats*) The *beating phenomenon* happens when two identical oscillators are "weakly coupled" to each other. The oscillators, in the notation of Figure 7.1.1, are identical if $k_1 = k_3 = \omega_0^2$. They are weakly coupled if[2] $k_2 = \epsilon$. The frequency ω_0 is the frequency both masses would have if there were no coupling ($\epsilon = 0$). Suppose only the first mass is "in action" at time $t = 0$, which means $x_2(0) = 0$ and $\dot{x}_2(0) = 0$. Then for a while x_1 moves much more than x_2. Most of the energy is in the first mass. Then, slowly, the energy "leaks" from x_1 to x_2. After a long time, x_2 has most of the energy and x_1 has almost none. Then the process reverses and the energy flows slowly back to x_1. The *beating* phenomenon is the slow periodic transfer of energy back and forth between x_1 and x_2. The beating frequency gets slower as the coupling gets weaker. There are many Youtube videos about this. Many involve the Wilberforce pendulum, in which the two modes involve two ways a mass can move hanging from a spring. Here is one of the better ones: Wilberforce pendulum.

(a) Show that when $\epsilon > 0$ there are two distinct frequencies, $\omega_1 < \omega_2$ separated by a small "detuning" $\Delta\omega = \omega_2 - \omega_1$. Find a formula for $\Delta\omega$.

(b) Give a "physical explanation" for the fact that $\omega_1 = \omega_0$. Why does the coupling spring not effect the motion in the mode where the masses move "together" (always in the same direction)? Give an explanation for the fact that adding a coupling spring increases the frequency of the mode where the masses move in opposite directions.

(c) Adapt the solution from Exercise 1(d) to write a formula in terms of cosines in this case.

(d) To interpret the formula from part (c), write[3] $\omega_1 = \overline{\omega} - \frac{1}{2}\Delta\omega$ and $\omega_2 = \overline{\omega} + \frac{1}{2}\Delta\omega$. Next, verify the formula $\cos(\theta - \delta) + \cos(\theta + \delta) = 2\cos(\delta)\cos(\theta)$. Next, write the formula for $x_1(t)$ in the form

$$x_1(t) = A(t)\cos(\overline{\omega}t) \quad , \quad A(t) = \cdots \ .$$

Use the formula for $A(t)$ to show that the amplitude of the x_1 oscillation slowly goes back and forth from its largest value to zero. This slow, long "time scale" behavior is the beating phenomenon.

(e) Show that $x_2(t)$ can be written in the form $x_2 = B(t)\cos(\overline{\omega}t)$ and show that B is large (positive or negative) when A is small and conversely.

3. (*Facts about permutations and determinants*) Suppose F represents some "operation" on something. For example, F could be a linear transformation $x \mapsto y = Ax$. Also, F could be a permutation $(1, 2, \cdots, n) \mapsto (\pi_1, \pi_2, \cdots, \pi_n)$. The *inverse* transformation is the transformation that undoes whatever F does. For example the inverse of a linear transformation is $y \mapsto x = A^{-1}y$. If you apply F and then F^{-1}, you get back what you started with. The inverse of a permutation puts the numbers back in their original order. That is, if $k = \pi_j$, then $j = \rho_k$. For example, the inverse of the permutation $\pi : (1, 2, 3, 4) \mapsto (2, 1, 4, 3)$ is ρ: $(1, 2, 3, 4) \mapsto (2, 1, 4, 3)$. Some checks: $\pi_2 = 1$ and $\rho_1 = 2$, $\pi_4 = 3$ and $\rho_3 = 4$. You should check this also works for π_1 and π_3. We write π^{-1} for the inverse permutation of π.

[2] Using ϵ to denote a number is a mathematician's way of saying the number is small.

[3] A bar over a variable can refer to the average value of that variable. Here, $\overline{\omega}$ is the average of ω_1 and ω_2.

(a) Show that π and π^{-1} have the same parity: $\text{sign}(\pi) = \text{sign}(\pi^{-1})$. *Hint.* If a sequence of transpositions makes π then the same transpositions make π^{-1} if done in the opposite order (why?).

(b) Show that if $\rho = \pi^{-1}$, then

$$\prod_{j=1}^{n} a_{j,\pi_j} = \prod_{k=1}^{n} a_{\rho_k,k} \; .$$

(c) The determinant is

$$\det(A) = \sum_{\pi} \left(\text{sign}(\pi) \prod_{j=1}^{n} a_{j,\pi_j} \right) \; . \tag{3}$$

Here, a_{jk} is the (j,k) entry of A. The sum is all $n!$ permutations. The (j,k) entry of A^T is a_{kj}. Use part (b) to show that $\det(A) = \det(A^T)$.

(d) Show that $\det(A)$ is "linear in the first column of A". This means, among other things, that if

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

and

$$B = \begin{pmatrix} b_{11} & a_{12} & \cdots & a_{1n} \\ b_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Then $\det(A+B) = \det(A) + \det(B)$

$$\det(A) + \det(B) = \det(A+B) = \det\left(\begin{pmatrix} a_{11}+b_{11} & a_{12} & \cdots & a_{1n} \\ a_{21}+b_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1}+b_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \right)$$

(e) Combine parts (c) and (d) to show that $\det(A)$ is also linear in the top row of A. Show that $\text{sign}(\pi') = -\text{sign}(\pi)$ and that the determinant changes sign if you interchange the first and second columns:

$$\det\left(\begin{pmatrix} a_{12} & a_{11} & \cdots & a_{1n} \\ a_{22} & a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n2} & a_{n1} & \cdots & a_{nn} \end{pmatrix} \right) = -\det\left(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \right)$$

(f) Let T be the transposition that interchanges elements 1 and 2 of n elements. Thus, $T(4,6,5,2,1,3) = (6,4,5,2,1,3)$. For any permutation π, let $\pi' = T\pi$, which is the same permutation as π but with the first and second elements switched. Show that $\text{sign}(\pi') = -\text{sign}(\pi)$ and that the determinant changes sign if you interchange the first and second columns:

$$\det\left(\begin{pmatrix} a_{12} & a_{11} & \cdots & a_{1n} \\ a_{22} & a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n2} & a_{n1} & \cdots & a_{nn} \end{pmatrix} \right) = -\det\left(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \right)$$

(g) Show that if the top two rows of A are identical then the determinant is zero

$$\det\left(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\right) = 0 \,.$$

(h) Combine the facts above to show that elementary row operations on a matrix do not change the determinant. It will be enough to do it for an operation involving the top two rows. If the row operation is subtracting c times the top row from the second row (a typical row operation) then you need to show that

$$\det\left(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} - ca_{11} & a_{22} - ca_{12} & \cdots & a_{2n} - ca_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\right) = \det\left(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\right)$$

(i) A matrix is *upper triangular* if all its entries below the diagonal are zero. We picture this by using $*$ to represent non-zero elements (elements that do not have to be zero).

$$\begin{pmatrix} * & * & * & \cdots & * & * \\ 0 & * & * & \cdots & * & * \\ 0 & 0 & * & & * & * \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & & \cdots & * & * \\ 0 & 0 & & \cdots & 0 & * \end{pmatrix}$$

In formulas, a matrix B is upper triangular if $b_{jk} = 0$ if $j > k$. Show that if B is upper triangular, then only the term with $\pi = \mathrm{id}$ in the sum (3) is different from zero. Here, "id" refers to the identity permutation $(1, 2, \cdots, n) \mapsto (1, 2, \cdots, n)$. Show that the determinant of an upper triangular matrix is equal to the product of its diagonal entries. *Comment.* Gauss elimination is a process that makes a general matrix A into an upper triangular matrix B using row operations. This exercise shows that you can compute the determinant of A by reducing it to upper triangular form and then multiplying the diagonal entries. Using the form (3) directly involves $n!$ terms, which makes it impractical for, say, $n = 30$. Using Gauss elimination, it is easy to evaluate the determinant of a 1000×1000 matrix.

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- Reviewing class notes
- Reading the textbook
- Reading supplementary notes
- Solving and writing up exercises
- Finally, if you found helpful (to you) online materials, please include a link or a URL.