Assignment 6, due March 10, 2pm

Correction: "$= \delta(t)$" and "$= f(t)$" $\rightarrow$ "$+\delta(t)$" and "$+f(t)$", and $f(x) \rightarrow f(s)$ in Exercise 1(b) part 2. The matrix A in Exercise 3 (4) completely replaced.

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- Solutions must be uploaded before class starts on the day assignments are due.
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1. (*Various approaches to linear ODE with forcing*) The ODE is

$$\ddot{x} = -5x - 4\dot{x} + 1 \ . \tag{1}$$

We want the solution with initial condition $x(0) = 0$, $\dot{x}(0) = 0$. This is the same as solving

$$\ddot{x} = -5x - 4\dot{x} + H(t) \ , \quad H(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t > 0 \ . \end{cases} \tag{2}$$

This forcing function that "turns on" at $t = 0$ is called the *Heaviside function*. The condition $x(t) = 0$ for $t < 0$ holds because the oscillator has friction so it doesn't move if something doesn't push it, and nothing pushes it before $t = 0$.

General plus particular solution approach

(a) The *general solution* is the vector space of solutions of the homogeneous problem $\ddot{x} = -5x - 4\dot{x}$ for all t. The *general solution* is the vector space of solutions of the homogeneous problem $\ddot{x} = -5x - 4\dot{x}$ for all t. Find two linearly independent exponential solutions (either in complex form involving complex exponentials or in real form involving real exponentials and sines and cosines) and express the general solution as a linear combination of these: $x_g(t) = au_1(t) + bu_2(t)$, with u_1 and u_2 being exponential solutions.

(b) A *particular solution* is any solution to the forced problem (1). Find a particular solution of the form $x_p(t) = Const.$

(c) Find a and b so that $x(t) = x_g(t) + x_p(t)$ satisfies the initial conditions given. Give the corresponding formula for $x(t)$.

Impulse response integral approach

(a) The *impulse response* is the solution of the initial value problem $\ddot{x} = -5x - 4\dot{x}$ for $t > 0$ with $x(0) = 0$ and $\dot{x}(0) = 1$. Show (by integration from $t = -\epsilon$ to $t = \epsilon$ and taking the limit $\epsilon \to 0$) that this is the solution to the ODE with "impulse" forcing

$$\ddot{x} = -5x - 4\dot{x} + \delta(t) \; ,$$

and $x = \dot{x} = 0$ for $t < 0$. Call this solution $R(t)$.

(b) Consider the general forced problem

$$\ddot{x} = -5x - 4\dot{x} + f(t) \; .$$

The solution is given by the impulse response integral

$$x(t) = \int_{-\infty}^{t} R(t-s) f(s)\, ds \; .$$

Use this to express the solution of (1) with the initial conditions given as an integral involving R. It might be helpful to use the Heaviside function formulation.

(c) "Work" the integral (find a closed form formula for it). If all the algebra is correct, this should agree with the solution by the "general plus particular solution" method.

2. (*Practice with the phase plane*) Consider the spring mass system with friction

$$m\ddot{y} = -ky - \gamma\dot{y} \; . \tag{3}$$

Reformulate this as a system of two first order differential equations in the form $\dot{x} = Ax$, $x(t) \in \mathbb{R}^2$. For each set of parameter values, make a large (about half a page or so) phase plane diagram showing the corresponding vector field. Use the vector field to determine the nature of the solution (periodic, spiral in/out, sink, saddle point, source, ... Determine whether there are real eigenvectors. If so, draw the corresponding lines through the origin and indicate by little arrows whether solutions along those eigen-lines flow toward or away from the origin. Identify the set of initial conditions so that $x(t) \to 0$ as $t \to \infty$.

(a) $m = 1$, $k = 2$, and $\gamma = 1$.

(b) $m = 1$, $k = 1$, and $\gamma = 4$.

(c) $m = 1$, $k = -1$, and $\gamma = 1$. *Comment.* An actual spring probably cannot have a negative spring constant, but the equation (3) models an unstable pendulum which is a mass on top of a rigid stick balancing on a point (as drawn in class).

3. (*Practice with eigenvalues and eigenvectors*) Consider the 3×3 system

$$\dot{x} = Ax \; , \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \; , \quad A = \begin{pmatrix} 5 & 3 & -1 \\ -6 & -4 & 2 \\ -1 & -1 & -1 \end{pmatrix} \; . \tag{4}$$

(a) Find the characteristic polynomial of A and its roots. *Hint.* If you find one root r_1 of a polynomial $p(r)$, you can *deflate* by computing $q(r) = p(r)/(r - r_1)$. This is the same as factoring $p(r) = (r - r_1)q(r)$. If p is cubic than q is quadratic, so you can use the quadratic formula on q. A cubic polynomial with real coefficients has at least one real root. Try to find one by figuring where p changes sign, by evaluating $p(r)$ at various r values. This trial-and-error method isn't pretty or quick, but it works.

(b) Find an eigenvector corresponding to each eigenvalue, $Av_j = r_j v_j$. Let V be the 3×3 matrix whose columns are the eigenvectors.

$$V = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} .$$

Compute $\det(V)$ to see whether V is invertible.

(c) Use the eigenvalues and eigenvectors to find first a complex basis (with complex exponentials) and then a real basis (with real vectors and sines and cosines) for $\mathcal{S}$, which is the vector space of all solutions of (4). The set of all elements of $\mathcal{S}$ is called the *general solution*.

(d) A *left eigenvector*[1] with eigenvalue r is a row vector $w = (a, b, c)$ so that $wA = rw$. Let W be the 3×3 matrix whose rows are the left eigenvectors:

$$W = \begin{pmatrix} - & w_1 & - \\ - & w_2 & - \\ - & w_3 & - \end{pmatrix} .$$

Find a left eigenvector for each eigenvalue and construct the matrix W. It is important that w_j and v_j correspond to the same eigenvalue r_j.

(e) Show that A is any $n \times n$ matrix and if w_j and v_k are left and right eigenvectors corresponding to different eigenvalues ($r_j \neq r_k$), then $w_j r_k = 0$. It is common in linear algebra to think of a row vector as a $1 \times n$ matrix and a column vector as an $n \times 1$ matrix. This makes product $w_j v_k$ a 1×1 matrix, which is a number. *Hint.* Matrix multiplication is associative, $(BC)D = B(CD)$, as long as the terms in the product have the right dimensions. Therefore $(w_j A)v_k = w_j(Av_))$, which gives a contradiction unless $w_j v_k = 0$ or $r_j = r_k$.

(f) Compute the matrix product $D = WV$. If all the arithmetic is correct, D is a diagonal matrix

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} .$$

Explain why the fact fro part (e) forces WV to be diagonal. Find $V^{-1} = D^{-1}W$. *Hint.* It is a theorem in linear algebra (a non-trivial theorem) that if C and B are $n \times n$ matrices with $CB = I$ then $BC = I$. Either way, $B = C^{-1}$.

(g) Let be the exponential solutions $u_j(t) = v_j e^{r_j t}$. Let $U(t)$ be the 3×3 matrix whose columns are the u_j:

$$U(t) = \begin{pmatrix} | & | & | \\ u_1(t) & u_2(t) & u_3(t) \\ | & | & | \end{pmatrix} .$$

Show that if M is any 3×3 matrix, then $P(t) = U(t)M$ satisfies the matrix differential equation

$$\dot{P} = AP .$$

Find a matrix M so that $P(0) = I$. This is the *fundamental solution*

(h) Carry out the algebra for this example and give an explicit expression for $P(t)$.

(i) Show by explicit calculation that $P(t + s) = P(t)P(s)$ for any times t and s. *Comment.* The solution to the scalar equation $\dot{p} = ap$ with $p(0) = 1$ is $p(t) = e^{at}$. This satisfies the equation $p(t + s) = p(t)p(s)$. The matrix fundamental solution $P(t)$, by analogy, is often written as a matrix exponential $P(t) = e^{tA}$. This exercise verifies the multiplicative property of this matrix exponential. But be careful, it is not true that $e^A e^B = e^{(A+B)}$ in general.

[1] Column vectors v with $Av = rv$ are *right eigenvectors* because the eigenvector is on the right of the matrix.