Differential Equations, Spring 2026
https://math.nyu.edu/~goodman/teaching/DifferentialEquations2026/DifferentialEquations.html

Assignment 8, due April 14, 2pm

About homework assignments

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- You may write assignments on paper then photograph or scan them. If you do that, please collect all the images into a single pdf file to upload. You may use handwriting or typing on a tablet and upload it in pdf format. You may use LaTeX, but this is not encouraged because LaTeX is less expressive than handwriting and (for me) takes longer to prepare.
- Solutions must be uploaded before class starts on the day assignments are due.
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- Please check the Brightspace forum corresponding to the assignment before you start working on the assignment and from time to time while you are working on it. There may be questions, comments, or (alas!) corrections that will help you.
- Please post any comments or questions or possible corrections on the Brightspace forum for the assignment.
- Please email the instructor director directly with personal matters including requests for a homework deadline extension.
- Be follow the **academic integrity policies** that apply to this class as explained on the **class web page**. In particular, do not submit solutions prepared by AI tools.

1. (*Matrix exponentials and power series, partial repeat*) The *matrix exponential* is defined using a power series

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \ . \tag{1}$$

We saw in class that the matrix exponential is the fundamental solution matrix for the ODE $\dot{x} = Ax$, which means that $x(t) = e^{tA}x(0)$. This follows from two facts: $e^{0A} = I$, $\frac{d}{dt}e^{tA} = Ae^{tA} = e^{tA}A$. We also saw in class that the "scalar" exponential series is *multiplicative*. That means that if a and b are numbers, then

$$e^{ta}e^{tb} = e^{t(a+b)}$$

$$\left(\sum_{n=0}^{\infty} \frac{t^n}{n} a^n \right) \left(\sum_{n=0}^{\infty} \frac{t^n}{n} b^n \right) = \left(\sum_{n=0}^{\infty} \frac{t^n}{n} (a+b)^n \right) \ .$$

(a) Determine whether this is necessarily true for matrices. Suppose A and B are $d \times d$ matrices. Multiply the matrix powers (compute the first few terms until you see the answer, or continue until you see the pattern) to see whether the corresponding matrix multiplication formula is true:

$$(??????) \qquad e^{tA}e^{tB} \stackrel{?}{=} e^{t(A+B)} \qquad (?????)$$

(b) A matrix is a 2×2 *Jordan block* with eigenvalue λ if it has the form

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} .$$

Find a formula for A^n and then e^{tA}. *Hint.* Compute the powers, first A^2, then A^3, and so on until you see the pattern. Verify the pattern by mathematical induction, using your formula for A^n and the formula $A^{n+1} = AA^n$. The entries of e^{tA} are power series given by sums involving the corresponding entries of A^n.

(c) A $d\times d$ matrix is diagonalizable if there is a matrix R with $L = R^{-1}$ so that

$$LAR = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_d) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & 0 & \ddots & \\ 0 & \cdots & & \lambda_d \end{pmatrix} .$$

Use the power series formula (1) (and some matrix linear algebra) to show that

$$e^{tA} = R\,\text{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \cdots, e^{\lambda_d t})\, L .$$

(d) A matrix function $M(t)$ is *uniformly bounded*, or *bounded uniformly* for all t, if there is a number C so that $|M_{jk}(t)| \leq C$ for every entry and every t. Show that if a matrix is diagonalizable with all eigenvalues pure imaginary, then its matrix exponential is uniformly bounded for all t. *Warnings.* There is no simple formula for C, just an explanation of why some C exists. The value of C depends on the dimension, d and on the entries in L and R. The entries of L and R do not have to be real even if A is real. *Hint.* If z_j are complex numbers, then $|z_1 + \cdots + z_d| \leq |z_1| + \cdots + |z_d|$. If u_j are complex numbers and ω_j are real, then $u_1 e^{i\omega_1} + \cdots$ is a uniformly bounded function of t (why?).

(e) Show that the matrix exponential may not be uniformly bounded for all t even if all the eigenvalues are pure imaginary if it is a Jordan block.

2. (*Neumann series*) The *Neumann series* formula is

$$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n . \tag{2}$$

This is a matrix version of the geometric series formula

$$\frac{1}{1-a} = 1 + a + a^2 + \cdots . \tag{3}$$

The geometric series formula is true (the sum on the right converges and is equal to the left side) if $|a| < 1$.

(a) Show that the Neumann series formula is true if the sum on the right converges absolutely. This happens, for example, if $\|A\| < 1$, because $\|A^n\| \leq \|A\|^n$. *Hint.* Multiply the Neumann series by $I - A$.

(b) Show that the Neumann series formula is true for the Jordan block matrix

$$B = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} .$$

This is true even though the corresponding A does not have $\|A\| < 1$ (if r is large). Verify the resulting formula for B^{-1} directly.

(c) The *unit square* in the plane, which we call S, is the set of points (x, y) with $0 \le x \le 1$ and $0 \le y \le 1$. The *image* of S under B is the set of points Bp for all points $p = \begin{pmatrix} x \\ y \end{pmatrix}$ in the unit square. Show that this image is a parallelogram and sketch the action of the mapping $p \mapsto Bp$ for some value of r (maybe $r \sim 1.7$ or $r \sim .4$, the value is not important). Sketch the action of B^{-1} on this parallelogram to see that it goes back to S.

(d) Euler used the geometric series formula (3) to "show" that

$$1 - 1 + 1 - 1 + \cdots = \frac{1}{2} \ .$$

Do you believe this? *Warning.* Please answer this question with at least one word even though it is not a serious question.

3. (*Radius of convergence*) Consider the Taylor series representation

$$f(x) = \frac{e^{\sin(x^2)}}{(4 + x^2)(5 + x^2)} = \sum_{n=0}^{\infty} a_n x^n \ .$$

Without determining the power series coefficients a_n, determine the radius of its convergence.

4. (*Local linear approximation, theory*) Discrete time dynamics take the form

$$x_{n+1} = f(x_n) \ . \tag{4}$$

Here, x_n is a vector with d components: $x_n \in \mathbb{R}^d$. This could be called a nonlinear vector recurrence relation or (as in the textbook, questionably) finite differences. The *orbit* of x_0 is the sequence $(x_0, x_1, \cdots)$. A *fixed point* is an x with $f(x_*) = x_*$. If $x_0 = x_*$ then $x_n = x_*$ for all $n > 0$. An orbit that starts at a fixed point stays at that fixed point. A fixed point is locally stable or unstable depending whether orbits that start with x_0 close to x_* converge to x_* or diverge away from it. Except in borderline cases, stability can be determined using local linear analysis using the Jacobian matrix $A = f'(x_*)$. The eigenvalue/eigenvector problem for A is to find v_j and λ_j with $Av_j = \lambda_j v_j$. As usual, the eigenvalues and eigenvectors need not be real even if A is real. For discrete dynamics, we write the eigenvalue in terms of its magnitude and phase $\lambda_j = r_j e^{i\theta_j}$, where $r \ge 0$ and θ are real.

(a) Suppose $\lambda = re^{i\theta}$ with $r = .9$ and $\theta = \frac{\pi}{10}$. Make a sketch of the orbit λ^n for $n = 0, 1, 2, \cdots$ in the complex plane. Draw enough points in the orbit to make the behavior of the orbit clear.

(b) Show that if A is real and $Av = \lambda v$, then $y_n = \text{Re}(\lambda^n v)$ is a real orbit for the linear recurrence $y_{n+1} = Ay_n$. That is, the real part of a complex orbit is a real orbit, as it is for linear ODEs.

(c) Suppose $d = 2$ and the real matrix A has eigenvalue $.9e^{i\pi/10}$ and corresponding eigenvector $v = \begin{pmatrix} i \\ 1 \end{pmatrix}$. Let y_n be the corresponding real orbit as in part (b). Make a sketch of the orbit in the real plane, showing y_0, y_1, $\cdots$. Draw enough points in the orbit to make the behavior of the orbit clear.

(d) Show, for any $d \ge 1$, that the fixed point $y_* = 0$ is stable if $|\lambda_j| < 1$ for all j. Show that the fixed point is unstable if $|\lambda_j| > 1$ for any j. Assume that A is diagonalizable. *Comment* A linear ODE system is stable or unstable depending on whether the eigenvalues are in the left or right half plane. For a linear recurrence, stability/instability depends on whether eigenvalues are inside or outside of the unit circle.

(e) Show that a fixed point for a non-linear recurrence is locally stable or unstable depending on whether eigenvalues of the Jacobian matrix are all inside or some outside the unit circle. Don't worry if you don't have the tools to make a rigorous proof. Just make an informal argument using that fact that if $x_n = x_* + y_n$, then $y_{n+1} \approx Ay_n$. Do your best at this, but do not spend too much time on it.

5. (*Local linear approximation, examples*) In each case, determine whether the fixed point x_* is (locally) stable or unstable using eigenvalue/eigenvector analysis and indicate which starting points x_0 lead to orbits that go toward or away from x_*. You will see that orbit diagrams for discrete time recurrences are not as neat or easy to draw as phase plane diagrams for ODEs. Do your best in a reasonable amount of time. It helps to find orbits that lie on or near a line in the plane, but there may not be any.

   (a) $x_{n+1} = Ax_n$ with
$$A = \begin{pmatrix} 1.3 & .4 \\ -.2 & \cdot 7 \end{pmatrix}$$

   (b) [1] $(x_{n+1}, y_{n+1}) = f(x_n, y_n)$ with
$$f(x,y) = \frac{1}{2}\begin{pmatrix} x^5(1+y^8) \\ (1+x^{-3})y^5 \end{pmatrix} \quad , \quad \begin{pmatrix} x_* \\ y_* \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

6. (*The $J_1(r)$ Bessel function*) The $J_1(r)$ Bessel function satisfies the ODE
$$J_1'' + \frac{1}{r}J_1' - \frac{1}{r^2}J_1 = -J_1 .$$
Assume a power series solution
$$J_1(r) = a_0 + a_1 r + a_2 r^2 + \cdots .$$

   (a) Show that $a_0 \neq 0$ is impossible by computing the behavior of the left and right sides as $r \to 0$.

   (b) Find a formula for a_2 in terms of a_0, a formula for a_3 in terms of a_1, for a_4 in terms of a_2.

   (c) Continue in the spirit of part (b) until you find a general formula for a_{k+2} in terms of a_k. *Comment.* Formulas of this type are *recurrence relations.*

   (d) Use the recurrence relation from part (b) and some ideas from Exercise **??** to show that the power series for $J_1(r)$ converges for all r, which is to say that the radius of convergence is infinite.

   (e) (**Do not not not try to do this**) Use power series calculations to show that (this is the *Kapteyn formula*)
$$\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k = 1 + 2\sum_{n=1}^{\infty} J_n(nr) .$$
   The functions on the right are Bessel functions of order n, which satisfy the ODE
$$J_n'' + \frac{1}{r}J_n' - \frac{n^2}{r^2}J_n = -J_n .$$

[1] This example is artificial. The numbers and formulas are chosen to make the computations simple. Real problems have more natural formulas, but the computations are impossible to do by hand.

The power series representation of $J_n(r)$ has all coefficients a_k with $k < n$ equal to zero and a specific conventional value for a_n:

$$J_n(r) = \frac{1}{2^n} r^n + \sum_{k>n} a_k r^k \ .$$

It should not be obvious that the infinite sum on the right converges, but the formula (if it's true) suggests that it should converge for $r < 1$. This non-exercise is here only to illustrate the fact that math has an endless supply of surprising formulas of the form:

$$\text{(one infinite series)} = \text{(another completely different looking infinite series)} \ .$$

7. Download and run the demo Python file `series_plot.py`. Adjust the code in lines 36 to 42 to implement the recurrence relations for the J_1 Bessel function, starting with $a_0 = 0$ and[2] $a_1 = \frac{1}{2}$. Adjust the comments and headers to indicate that you modified code you downloaded (your contact information, date, and purpose) and that it's for `J_1` rather than cosine. Produce two plots and a few sentences describing the results

   (a) A plot similar to the cosine plot showing how various (not very large) values of n approximate $J_1(r)$ for different ranges of r. The range of r for the plot should include a few oscillations of J_1, but not a large number. The largest n value should be large enough that the partial sum is very close to J_1 in that range. Be careful to increase `n_max` if you need to.

   (b) Take a value of n large enough to see the values of at least the first five r_j where $J_1(r_j) = 0$ (not counting $r = 0$ even though $J_1(0) = 0$). The internet thinks the first three are $r_1 \approx 3.8$, $r_2 \approx 7.0$, and $r_3 \approx 10.2$. *Comment.* These "Bessel function zeros" determine frequencies of vibration for the 2D wave equation in a circular domain. *Comment.* If you take Numerical Analysis, you will learn computational methods such as Newton's method that can find the Bessel zeros accurately once J_1 and J_1' are evaluated accurately. In this exercise, just estimate the zeros by looking at where the J_1 curve crosses the axis.

[2] The goal of the computation is to find the Bessel zeros, which don't depend on the value of a_1, except that it is not zero. The value $a_1 = \frac{1}{2}$ is a convention.