Assignment 9, due April 21, 2pm

About homework assignments

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- Solutions must be uploaded before class starts on the day assignments are due.
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- Please check the Brightspace forum corresponding to the assignment before you start working on the assignment and from time to time while you are working on it. There may be questions, comments, or (alas!) corrections that will help you.
- Please post any comments or questions or possible corrections on the Brightspace forum for the assignment.
- Please email the instructor director directly with personal matters including requests for a homework deadline extension.
- Be follow the **academic integrity policies** that apply to this class as explained on the **class web page**. In particular, do not submit solutions prepared by AI tools.

Codes and plots

All the codes and plots for this assignment are posted with the assignment on the assignments page.

The exercises

1. (*Power series calculations, compound interest*) If r is an interest rate (a percentage per time period), then in one time period an account A gets rA added. The result is $A \to (1+r)A$. Over two interest rate periods, the result is $A \to (1+r)A \to (1+r)^2A$. If you just add rA each period, the result is $A \to (1+2r)A$. The difference is $(1+r)^2 - (1+2r) = r^2$. If r is small, the difference between simple $(1+2r)$ and compound $(1+r)^2$ is very small. For example, if r is $5\% = .05$, then $r^2 = \left(5\cdot 10^{-2}\right)^2 = 25\cdot 10^{-4} = .0025$. This is 20 times smaller than r.

   Suppose you replace one period with r with n periods with rate $\frac{1}{n}r$. Then the compound interest formula is
   $$M_n = \left(1 + \frac{r}{n}\right)^n \ .$$
   In calculus, you learn that $M_n \to e^r$ as $n \to \infty$. This can be quantified to determine the difference between M_n and e^r. For this, assume a power series ansatz
   $$M_n = a_0 + a_1\frac{1}{n} + a_2\frac{1}{n^2} + \cdots \ .$$

This is a power series in the small parameter $x = \frac{1}{n}$. Calculate a_0, a_1, and a_2. You can do this by calculating the expansion of $A_n = \log(M_n)$ using the Taylor series $\log(1+\epsilon) = \epsilon - \frac{1}{2}\epsilon^2 + \cdots$ (you need to figure out how many terms you need), You can then take $M_n = e^{A_n}$, use a Taylor expansion $e^{r+\epsilon} = e^r e^\epsilon = e^r(1+\epsilon+\cdots)$.

2. (*Error growth*) Consider the initial value problem

$$\begin{aligned} \dot{x} &= \ \ y \\ \dot{y} &= -x \\ x(0) = 1 \ \ &, \quad y(0) = 0 \ . \end{aligned}$$

The code formulates this in matrix/vector form as

$$\dot{x} = Ax \ \ , \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ \ , \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ \ , \quad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ . \tag{1}$$

We solve it using the forward Euler method with time step h. The approximate solution values are $x_n^h \approx x(t_n)$, $y_n^h \approx y(t_n)$, with $t_n = nh$.

   (a) (*review*) Find the solution of the initial value problem (1). Show that the solution trajectory is a circle with one orbit every 2π units of time.

   (b) (*review*) Show the solution of $\dot{x} = Ax$, with this A and any initial condition, has (using the notation of the scalar (x, y) formulation)

$$\frac{d}{dt}R^2 = \frac{d}{dt}\left(\, x(t)^2 + y(t)^2 \,\right) = 0 \ \ , \qquad \text{so } R(t) = R(0) \ . \tag{2}$$

   This shows that any trajectory is a circle.

   (c) The numerical radius is

$$R_n^h = \sqrt{\left(x_n^h\right)^2 + \left(y_n^h\right)^2} \ .$$

   The trajectory plot in `Euler_circles.pdf` shows that R_n^h is not a constant, but grows with n. This growth is slower when h is small, but is never exactly zero as (2). Find an approximate formula for the (false) growth per time step $G_n^h = R_{n+1}^h - R_n^h$. This formula has G being proportional to something involving R_n^h and some power of h. Do just enough algebra to find the "leading term", which is the term with the smallest power of h.

   (d) This growth is like compound interest, in that $R_{n+1}^h = (1+\epsilon)R_n^h = (1+\epsilon)^2 R_{n-1}^h = \cdots$. Find the (approximate) doubling time, which is the value of T as a function of h so that $R_n^h \approx 2$. The relations are, as always in time stepping solution, $T = nh$. This T should go to infinity as h goes to zero. The problem is to find out how fast. It might be $T \sim h^{-1}$ or $T \sim \log\left(h^{-1}\right)$. These are very different answers (at most one and possibly neither being correct). In the slow growth scenario ($T \sim h^{-1}$), replacing h by $\frac{1}{2}h$ makes the circle grow half as fast. In the fast growth scenario, replacing h by $\frac{1}{2}h$ makes the circle grow only a little bit slower.

   (e) Download the code `Euler_circles.py` and play with parameters to verify your approximate formula from part (d). Choose various h values and see what T gives $R \approx 2$. Do not look for high precision. What's important is the approximate size of T (within, say, 20%), and how T scales with h.

3. (*saddle points*) A fixed point for an ODE is a *saddle point* if the eigenvalues of the linearization are real and have both signs (at least one positive and at least one negative). The code `Euler_saddle.py` computes some trajectories for a linear saddle point ODE. Notice that trajectories that start near the "stable" eigenvector direction go close to the fixed point (which is the origin in this linear example) before moving away along the unstable eigenvector direction.

(a) Explain this phenomenon using the eigenvalue/eigenvector solution formula for a linear ODE system.

(b) Modify the matrix A in the code to make the stable and unstable directions less perpendicular. Choose a new family of starting points x_0 adapted to the new problem that illustrate the near stability/instability phenomenon. *Hints.* A 2×2 matrix A has real eigenvalues of opposite sign if $\det(A) < 0$ (If the eigenvalues have the same sign or if they are complex conjugate pairs, the determinant is positive, why?). If the trace (the sum of the diagonals is also the sum of the eigenvalues) is negative, then the negative eigenvalue is "stronger" than the positive one so trajectories starting near the unstable direction can get closer to the origin (why?). Increasing the parameter `T` makes longer trajectories. The code takes a noticeable amount of time to run on my newish laptop. If it's too slow for you, reduce the number of time steps, but be careful that the trajectories are still reasonably accurate. Notice that the code prints the eigenvalues and eigeenvectors.