Supplementary notes

Blow-up

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There are nonlinear differential equations that have solutions that *blow up* at a finite *blow up time*, $T < \infty$. This means that $x(t)$ is defined for $t < T$ and

$$x(t) \to \infty \quad \text{or} \;\; x(t) \to -\infty \quad \text{as } t \to T \; . \tag{1}$$

Suppose $x(t)$ satisfies the differential equation

$$\dot{x} = \frac{dx}{dt} = f(x) \; . \tag{2}$$

If this has a solution with the blow-up property (1), then we say (2) *allows blow-up*. If a differential equation allows blow-up, it might be that some solutions blow up and others do not. We know that linear differential equations do not allow blow-up. The solution formula in the book shows that $x(t)$, for a linear equation, is defined and finite for all t.

One differential equation that allows blow up is

$$\frac{dx}{dt} = x^2 \; . \tag{3}$$

This is an example of a *Ricatti* equation.[1] Suppose we start with initial condition

$$x(0) = x_0 = \frac{1}{3} \; . \tag{4}$$

The solution formula is

$$x(t) = \frac{1}{3-t} \; . \tag{5}$$

This formula satisfies the initial condition (4). The formula also satisfies the differential equation, because

$$\frac{d}{dt}\,\frac{1}{3-t} = \frac{1}{(3-t)^2} = \left(\frac{1}{3-t}\right)^2 \; .$$

Despite these calculus checks, the formula (5) is not always the solution to the *initial value problem* (2)(4).

You can see that something is wrong with the solution formula (5)by looking at whether $x(t)$ is positive or negative. The right side of (3) is positive, which means that $x(t)$ is an increasing function of t. Starting at $x_0 = \frac{1}{3}$ at $t = 0$, this implies that $x(t) > \frac{1}{3}$ if $t > 0$. The value $x(2) = 1$ is consistent with this. However, is you put $t = 4$ into the *supposed* solution formula (5), you get $x(4) = -1$. Rather than increasing from $x_0 = \frac{1}{3}$, the solution has decreased and even become negative. This violates a basic fact of calculus: the function is increasing if its derivative is positive.

[1] *Ricatti equations* are differential equations where $f(x)$ is a quadratic polynomial. Later in the course we will see vector and even matrix Ricatti equations. They come up in problems of *optimal control*.

What goes wrong is that the solution (5) blows up at blow-up time $T = 3$. The solution is positive (finite) and increasing for $t < T = 3$, but $x(t) \to \infty$ as $t \to T$. This may be expressed as

$$\lim_{t \uparrow T} x(t) = \infty \ . \tag{6}$$

The upward arrow in the limit means that t goes to T through values $t < T$. We say t approaches T "from below". It is natural to think there is no solution beyond the blow-up time. Even though the formula (5) makes sense for $t > T$, it does not represent the solution of the *initial value problem* (2) and (4). A *local in time* solution, like this one, exists for a finite range of t. A *global in time* solution exists for all $t > 0$. Some differential equation solutions are *local* while others are *global*.

A differential equation that allows blow-up may not require all solutions to blow up. For example, take the differential equation (3) with initial condition

$$x_0 = -\frac{1}{2} \ . \tag{7}$$

The solution, as is easily checked, is

$$x(t) = \frac{-1}{t+2} \ . \tag{8}$$

This solution formula is valid for all $t > 0$. The global in time solution increases and approaches 0 as $t \to \infty$.

What is the physical interpretation of a solution that blows up? It is that the differential equation (2) becomes invalid or inaccurate as a model before time T. For initial condition $x_0 = \frac{1}{3}$, the differential equation model (3) predicts that $x(t)$ becomes very large at t approaches T. However, the physical thing that x models cannot literally become infinite. Thus, the model only predicts that $x(t)$ "blows up" in the sense of becoming so large at time T that the model no longer applies. The model predicts that if x starts at $x_0 = -\frac{1}{2}$ then it does not blow up in this sense.

Exercises

1. Consider the differential equations

$$\text{(A)} \qquad \frac{dx}{dt} = x^2 - x$$

$$\text{(B)} \qquad \frac{dx}{dt} = x^2 + x$$

$$\text{(C)} \qquad \frac{dx}{dt} = x^2 + 1$$

   In each case, figure out which initial values x_0 lead to finite time blow-up and which have global in time solutions. *Hint.* You do not need to write solution formulas to figure this out. This is good because it takes a while to find the solutions in each case. Instead, think about where the solution is increasing, whether it can get to a large positive value, whether it can cross $x = 0$, what happens if it gets to a large positive value, etc.