Supplementary notes

Complex exponentials

Complex exponentials, Euler's formula

If z is any number, then the exponential may be defined by the infinite series[1]

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \ . \tag{1}$$

We will not stop here to explain why the sum converges or why it satisfies the addition formula

$$e^z e^w = e^{z+w} \ . \tag{2}$$

We will return to these topics later in the class when we discuss infinite series solutions to differential equations.

We follow Euler's suggestion to use i to represent an "imaginary" number with $i^2 = -1$ The letter "i" is for "imaginary", because no actual number x has a negative square. Any complex number z may be written in terms of its real part, x, and its imaginary part y as

$$z = x + iy \ .$$

The real and imaginary parts of a complex number, which themselves are real numbers, are written

$$x = \mathrm{Re}(z)$$
$$y = \mathrm{Im}(z) \ .$$

This z is called *real* if its imaginary part is zero. We sometimes say a number is "complex" if it is not real, but this is not strictly correct because real numbers also count as complex numbers. Complex numbers are not "real" (unless the imaginary part is zero), but they satisfy the arithmetic properties of real numbers (addition, subtraction, multiplication, division by non-zero numbers, associativity, distributivity, commutativity, etc.). The "size" of a complex number, which may be called its *norm*, is

$$|z| = \sqrt{x^2 + y^2} \ .$$

A sequence of complex numbers may have a limit, which would be a complex number. An infinite sum of complex numbers, such as (1), may converge. If it does converge, the infinite sum is also a complex number. Thus, if z is a complex number, then e^z is also a complex number. It is possible that e^z is real even though z is not real.

Euler's formula[2] is

$$e^{i\theta} = \cos\theta + i\sin(\theta) \ . \tag{3}$$

[1] There are other ways to define exponentials other than using the sum (1). One other way is $e^z = \lim_{n\to\infty}\left(1 + \frac{z}{n}\right)^n$. Another is to define $\log(y) = \int_1^y \frac{1}{u}du$ and then define e^x as the inverse function of log. That is, $e^x = y$ is equivalent to $\log(y) = x$. The sum formula (1) and the limit formula have the advantage that they make sense even when z is a complex number.

[2] Euler was a prolific early explorer of mathematical formulas and discovered many. This is just one of the dozens of mathematical facts called "Euler's formula".

We suppose θ is real, because the natural definitions of sine and cosine involve drawings in the (real) circle. If z is a complex number, then we can use addition formula (2) and Euler's formula (3) to give another definition of e^z:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x \left(\cos y + i \sin(y) \right) \ . \tag{4}$$

Thus

$$\mathrm{Re}\left(e^{x+iy} \right) = e^x \cos(y) \ , \quad \mathrm{Im}\left(e^{x+iy} \right) = e^x \sin(y) \ .$$

You can take (4) to be the definition of the complex exponential in terms of the real exponential and the real trig functions.

Professional mathematicians often use Euler's formula to re-derive angle sum formulas from trigonometry that they cannot remember. The idea uses the product formula for complex numbers

$$(a+ib)(c+id) = ac + iac + ibc + i^2cd = ab - cd + i(ac+bd) \ .$$

We apply this with Euler's formula (3) to get, on the one hand

$$e^{i(\theta+\phi)} = \cos(\theta+\phi) + i\sin(\theta+\phi)$$

and, on the other hand, the addition formula (2) implies that

$$\begin{aligned} e^{i(\theta+\phi)} &= e^{i\theta}\, e^{i\phi} \\ &= \left[\cos(\theta) + i\sin(\theta) \right] \cdot \left[\cos(\phi) + i\sin(\phi) \right] \\ &= \left[\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) \right] + i \left[\cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi) \right] \end{aligned}$$

Now, set the real and imaginary parts equal and you get the trig identities

$$\begin{aligned} \cos(\theta+\phi) &= \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) \\ \sin(\theta+\phi) &= \cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi) \ . \end{aligned}$$

The minus sign in the cosine formula is seen to come from $i^2 = -1$, One thing to see here is that working with complex exponentials rather than with real sines and cosines can make algebra simpler. You don't have to do it that way if you're uncomfortable with complex exponentials, but it's quicker and less error-prone if you do.

Derivatives of complex exponentials

Exponentials come up in solutions of differential equations because they satisfy the differential equation

$$\frac{d}{dx} e^x = e^x \ . \tag{5}$$

You can see this using the series definition (1). The derivative of a sum is the sum of the derivatives, so[3]

$$\frac{d}{dx} e^x = \sum_{n=0}^{\infty} \frac{d}{dx} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d}{dx} x^n \ . \tag{6}$$

A typical term in the sum on the right may be understood as

$$\frac{1}{n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1} \times n x^{n-1} = \frac{1}{(n-1) \cdot \cdots \cdot 2 \cdot 1} \times x^{n-1} = \frac{x^{n-1}}{(n-1)!} \ .$$

[3]To be mathematically correct, we only know the derivative of a sum formula for finite sums. The proof for infinite sums might actually depend on the sum. We ignore this possibility for now, but return to it later when we discuss infinite sum formulas for differential equations later in the course.

Also, the $n = 0$ term in the sum on the right of (6) is

$$\frac{d}{dx} 1 = 0 \ .$$

Therefore, (6) may be rewritten without the $n = 0$ term:

$$\frac{d}{dx} e^x = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \ .$$

The sum on the right is the sum in (1). Both may be written out as

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \ .$$

This proves the exponential derivative formula (5).

The chain rule applied to the differentiation formula leads to

$$\frac{d}{dx} e^{f(x)} = f'(x) \, e^{f(x)} \ .$$

For example, taking $f(x) = x^2$ gives

$$\frac{d}{dx} e^{x^2} = 2x e^{x^2} \ .$$

Even simpler, if a is a constant and $f(x) = ax$, we get

$$\frac{d}{dx} e^{ax} = a e^{ax} \ . \qquad (7)$$

This formula is true even if a is a complex number. If this makes you nervous, you can go through the derivation and see that every step works even if a is not real. Alternatively, you can write the complex a in terms of its real and imaginary parts as $a = b + ic$ and use Euler's formula (4), together with $ax = (b + ic)x = bx + icx$ to identify the real and imaginary parts of ax. Then you can calculate using the product rule and the formula for multiplying complex numbers:

$$\begin{aligned}
\frac{d}{dx} e^{ax} &= \frac{d}{dx} e^{(b+ic)x} \\
&= \frac{d}{dx} e^{bx+icx} \\
&= \frac{d}{dx} \left(e^{bx} \cos(cx) + i e^{bx} \sin cx \right) \\
&= \frac{d}{dx} e^{bx} \cos(cx) + i \frac{d}{dx} e^{bx} \sin cx \\
&= b e^{bx} \cos(cx) - c e^{bx} \sin(cx) + i b e^{bx} \sin(cx) + i c e^{bx} \cos(cx) \\
&= (b + ic) e^{bx} \left(\cos(cx) + i \sin(cx) \right) \\
&= a e^{ax} \ .
\end{aligned}$$

This long calculation should re-enforce the idea that working with and trusting complex exponentials directly can be simpler than using sines and cosines.

Real exponentials e^{bx} describe growth (if $b > 0$) or decay (if $b < 0$). Complex exponentials of the form e^{bx+icx} describe growth or decay *together with oscillation*. *Oscillation* means "go up and down", which is what sines and cosines do. The function $f(x) = e^{ax}$ has a growth rate given by the real part of a and an oscillation frequency given by the imaginary part

$$e^{ax} = e^{bx} \cos(cx) + i e^{bx} \sin(cx) \ .$$

The real part, $e^{bx}\cos(cx)$, and the imaginary part, $e^{bx}\sin(cx)$, have the same growth or decay rate and the same frequency. They are "out of phase", in that one involves the cosine while the other involves the sine. The size of the complex exponential is a real number that grows or decays with rate b

$$|e^{ax}| = \sqrt{(\,e^{bx}\cos(cx)\,)^2 + (\,e^{bx}\sin(cx))^2} \;=\; e^{bx}\ .$$

You can think of the real and imaginary parts of a complex number as coordinates in the *complex plane.*

$$z = x + iy \iff (x,y) \in \mathbb{R}^2\ .$$

The *right half plane* is the set of complex numbers whose real part is positive. The *left half plane* corresponds to negative real part. The behavior of the complex exponential e^{ax} depends on whether a is in the left or right half planes. If a is in the left half plane, then $b < 0$ and the exponential function is decreasing. If a is in the right half plane, then the exponential is an increasing function of x. The borderline case is $b = 0$, which means a is "on the *imaginary axis*". In that case the complex exponential is "purely oscillatory" with no growth or decay. You can distinguish between the *open* left half plane, which corresponds to negative real part, and the *closed* half plane, which corresponds to non-positive real part. If $b = 0$ then a is in the closed left half plane but not in the open left half plane. The terms *open* and *closed* come from mathematical analysis. The imaginary axis is the *boundary* of the left half plane (open or closed) and also of the right half plane. In mathematical analysis, a set is called *closed* if it includes its boundary. A set is called *open* if its *complement* (the points not in the set) is closed. Thus, the complement of the open left half plane is the closed right half plane, and so on. Complex exponentials are called *stable* or *unstable* depending on whether they grow or decay. A stable exponential might be "strictly stable" if a is in the open left half plane and "weakly stable" if a is on the imaginary axis.

Integrating with complex exponentials

Integrals involving exponentials and sines or cosines can be evaluated using the differentiation formula for complex exponentials together with Euler's relation (4). The complex exponential derivative formula (7) is equivalent to an integral formula

$$\int_0^{x_1} e^{ax}\,dx = \frac{1}{a}\,(\,e^{ax_1} - 1\,)\ . \tag{8}$$

Both sides of this formula are complex numbers, so the real parts and imaginary parts must be equal separately. In particular

$$\text{Re}\left[\int_0^{x_1} e^{ax}\,dx\right] = \text{Re}\left[\frac{1}{a}\,(\,e^{ax_1} - 1\,)\right]\ . \tag{9}$$

Suppose $a = b + ic$ (with b and c real). Then

$$\begin{aligned}
\text{Re}\left[\int_0^{x_1} e^{ax}\,dx\right] &= \text{Re}\left[\int_0^{x_1} e^{bx}\,(\,\cos(cx) + i\sin(cx)\,)\;dx\right] \\
&= \int_0^{x_1} e^{bx}\cos(cx)\,dx \\
&= \text{Re}\left[\frac{1}{b+ic}\;\left\{\,e^{bx_1}\,(\,\cos(cx_1) + i\sin(cx_1)\,) - 1\right\}\right] \\
&= \text{Re}\left[\frac{1}{b+ic}\;\left\{\,\left(e^{bx_1}\cos(cx_1) - 1\right) + ie^{bx_1}\sin(cx_i)\,\right\}\right]
\end{aligned}$$

We can calculate this using the algebra of complex numbers. First is the old trick,

$$\frac{1}{b+ic} = \frac{b-ic}{b-ic}\,\frac{1}{b+ic} = \frac{b-ic}{b^2+c^2}\;.$$

This gives the integral as

$$\begin{aligned}\int_0^{x_1} e^{bx}\cos(cx)\,dx &= \mathrm{Re}\left[\,\frac{b-ic}{b^2+c^2}\;\left\{\,\left(e^{bx_1}\cos(cx_1)-1\right)+ie^{bx_1}\sin(cx_i)\,\right\}\,\right] \\ &= \frac{1}{b^2+c^2}\,\mathrm{Re}\left[\,(b-ic)\;\left\{\,\left(e^{bx_1}\cos(cx_1)-1\right)+ie^{bx_1}\sin(cx_1)\,\right\}\,\right] \\ \int_0^{x_1} e^{bx}\cos(cx)\,dx &= \frac{1}{b^2+c^2}\;\left[\,b\left(e^{bx_1}\cos(cx_1)-1\right)-ce^{bx_1}\sin(cx_1)\,\right]\;. \end{aligned} \tag{10}$$

This gives the real formula for the real integral involving an exponential and a cosine. The derivation is not so quick, but is does not require any cleverness. You can find the same formula in a calculus book, where it is done using integration by parts.

You can do some checks on (10). For example, the answer is supposed to be 0 if $x_1 = 0$. You can check that the right side of (10) is 0 when $x_1 = 0$. You can also check that it simplifies to the right answer if $b = 0$ or $c = 0$.

Calculus books have something they call the *indefinite integral;*. They write

$$\int f(x)\,dx = g(x) + C \tag{11}$$

to mean that

$$g'(x) = f(x)\;. \tag{12}$$

Actual integrals have limits of integration. The indefinite integral formula is a step toward doing actual integrals with actual limits of integration

$$\int_{x_0}^{x_1} f(x)\,dx = g(x_1) - g(x_1)\;. \tag{13}$$

Calculus books call actual integrals *definite* integrals. The formula (10) is about actual integrals. The indefinite integral version starts with the differentiation formula (7) in the form

$$\int ae^{ax}\,dx = e^{ax} + C\;.$$

Then you divide both sides by a and write C for $\frac{1}{a}C$ to get

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C\;.$$

Next, take the real part of both sides to get (with yet another C)

$$\int e^{bx}\cos(cs)\,dx = \mathrm{Re}\left(\frac{1}{a}e^{ax}\right) + C\;.$$

Be careful not to confuse "little c" (the imaginary part of a) with "big C" (the supposed "constant of integration, whatever that means). Finally, as before,

$$\begin{aligned}\mathrm{Re}\left(\frac{1}{a}e^{ax}\right) &= \frac{e^{bx}}{b^2+c^2}\mathrm{Re}\left((b-ic)\left[\,\cos(cx)+i\sin(cx)\,\right]\right) \\ &= \frac{e^{bx}}{b^2+c^2}\left[\,b\cos(cx)-c\sin(cx)\,\right]\;.\end{aligned}$$

The indefinite integral version of (10) is

$$\int e^{bx} \cos(cx)\, dx = \frac{e^{bx}}{b^2 + c^2} \left[\, b\cos(cx) - c\sin(cx)\, \right] + C \ .$$

This formula is slightly simpler than the definite integral formula (10). It is a way of saying that

$$\frac{d}{dx} \frac{e^{bx}}{b^2 + c^2} \left[\, b\cos(cx) - c\sin(cx)\, \right] = e^{bx} \cos(cx) \ .$$

You can check this by differentiation.

Finally, to make a connection with the notes on differentials, observe that you can write the differentiation relation (12) as

$$\frac{dg}{dx} = f(x) \ .$$

If you multiply both sides by dx, this becomes

$$dg = f(x)\, dx \ . \tag{14}$$

Then, you can add up all the small changes in g between x_0 and x_1 to get the total change in g between these x values:

$$\int_{x_0}^{x_1} dg = g(x_1) - g(x_0) \ .$$

The differential relation (14) then implies that the definite integral relation (13). To summarize, the definite integral represents the sum of many small changes in some quantities in some range. The indefinite integral is a convenient notation for differentiation. The relation between them is called the *fundamental theorem of calculus*, which is the statement that (a) when you add up all the dg changes, you get the total change, and (b) the difference between dg and $f(x)dx$ is "tiny" in the sense of the notes on differentials.