Supplementary notes

Differentials

Reasoning with differentials

Most people reason informally using *differentials* when they work with differential equation models. You need to understand this kind of reasoning, even if you don't like it, because that's the way most people outside math departments think. To be concrete, consider the differential equation

$$\frac{dx}{dt} = x^2 \ . \tag{1}$$

For this example, we take *initial data*

$$x(0) = \frac{1}{3} \ . \tag{2}$$

We want to "solve" this *initial value problem* by writing a formula for $x(t)$:

$$x(t) = \frac{1}{3-t} \ . \tag{3}$$

Although no derivation is given, you can check that the solution formula is correct. Differentiating (13) with respect to t shows that it satisfies (1). Taking $t = 0$ in (3) gives (2). This solution formula is an any differential equations textbook. We now give a derivation using the informal language and reasoning of differentials.

First, you think of $\frac{dx}{dt}$ as a fraction and multiply both sides of (1) by dt. This gives

$$dx = x^2 \, dt \ . \tag{4}$$

You then "separate variables" by dividing both sides by x^2, which leads to

$$\frac{dx}{x^2} = dt \ . \tag{5}$$

Then you "integrate" both sides. The left side is the differential of $-\frac{1}{x}$:

$$\frac{dx}{x^2} = d\left(-\frac{1}{x}\right) \ .$$

The right side is the differential of t. Thus (5) is equivalent to

$$d\left(-\frac{1}{x}\right) = dt \ . \tag{6}$$

If the differentials of two quantities are equal, then the quantities themselves must be "equal up to a constant". The difference between them cannot change (is constant). Thus, the differential relation (6) implies that there is some constant C so that

$$-\frac{1}{x} = t + C \ . \tag{7}$$

Finally, you want to get $x = \frac{1}{3}$ when $t = 0$, so you put those values into the "integrated" relation (7) and find

$$-3 = C \; .$$

This value of C turns (7) into

$$-\frac{1}{x} = t - 3 \; .$$

This is equivalent to our solution formula (3).

This reasoning is not rigorous, partly because differentials like dt or $d(x^{-1})$ are not precisely defined.[1] Informally, if Q is a quantity or expression, then the informal dQ means "the small change in Q. If Q and R are related quantities and something happens to change both of them, then it is common that the changes are approximately proportional:

$$dQ \approx C\, dR \; . \tag{8}$$

For example, suppose Q is a measure of the color of liquid and dR is one drop of die (a small amount). Then dQ measures how much the color changed, which also is small. It is natural that if you use two drops instead of one then the color changes twice as much. This might be expressed as

$$(\, dR \rightarrow 2dR \,) \implies (dQ \rightarrow 2dQ \,) \; .$$

That is, if dR is small, then the change in Q is (approximately) proportional to the change in R. The constant of (approximate) proportionality may be expressed as a ratio

$$\frac{dQ}{dR} \approx C \; .$$

The informal reasoning with differentials can be thought of as writing = for ≈. Suppose L and M are two quantities that are close enough that it does not matter, for some purpose, whether you use L or M. Then you might write $L = M$ even though they are not exactly equal. For example, you could write $\pi = 3.14$ to mean that if you use 3.14 instead of the exact value $3.14315926535\cdots$, the result is "the same" (the difference doesn't matter). For example, you might be calculating the area of a circular tabletop from its radius ($A = \pi r^2$) and an error of less than 1% is too small to matter.

With differentials, the linear approximation (8) might be so accurate that it doesn't matter whether you use dQ or $C\, dR$ in some formula. In this spirit, the formula (4) might be taken to mean that if dt and dx are small enough then the difference between $x^2 dt$ and dx does not matter. The difference in the x values, which we call dx, is (almost) the same as x^2 times the difference in the t values, which is dt. One mathematician[2] joked that you have to distinguish between "small" and "tiny". Small differences like dx and dt matter, but tiny differences, like $dx - x^2 dt$ don't matter We put an = sign between small terms when the difference between them is tiny.

Let us return to the the point made before that if two quantities have the same differentials then they are equal, "up to a constant" (the difference between then does not change). The process of adding together many small differential changes to get the overall change is *integration*. Suppose the numbers dt_k form a sequence of small changes in t, starting at t_s and ending at at a final time t_f. Let the numbers dx_k be the corresponding sequence of small x increments. These can be added up

[1] The pure math subject of *differential geometry* has "differentials", also written with a d, but which mean something different. There are calculus books that say, for example, $du = u'(x)dx$ is the *definition* of du. These books claim this is the basis for the change of variables formulas in integrals such as: $\int f\, du = \int f u'(x)\, dx$. This is nonsense. Here, df just means "a small change in f", the same for dx, dt, etc.

[2] I believe it was Sir Michael Atiyah, winner of the Fields Medal, which is the mathematicians' equivalent of a Nobel Prize.

to get[3]

$$t_f - t_s = \sum_k dt_k \;\rightarrow \int dt$$
$$x_f - x_s = \sum_k dx_k \rightarrow \int dx \; .$$

These express the express the idea that you get the total change in some quantity between a starting point and a final point by adding together a sequence of small changes along a path from start to final. The informal relation (6) says that the difference between $d\left(-x^{-1}\right)$ and dt is "tiny" in the sense that it is so small that even when you add up all the little errors for each k, the total error goes to zero in the limit $dt \rightarrow 0$ and $dx \rightarrow 0$. So, if we write

$$d\left(\frac{1}{x}\right)_k$$

for the change in $\frac{1}{x}$ in the time interval dt_k, then (6) means that

$$-\sum_k d\left(\frac{1}{x}\right)_k = \sum_k dt_k \; . \tag{9}$$

The two sides in this informal "equation" are not exactly equal, but they "become equal" as the increments dt_k and dx_k become infinitely small. The sum of differences on the left side of (9) becomes, or "is" the total difference

$$-\frac{1}{x(t_f)} - \left[-\frac{1}{x(t_s)}\right] \; .$$

The sum of differences on the right side is $t_f - t_s$. Therefore

$$-\frac{1}{x(t_f)} - \left[-\frac{1}{x(t_s)}\right] = t_f - t_s \; . \tag{10}$$

Now, write t for t_f and keep the starting time t_s fixed. This leads to

$$-\frac{1}{x(t)} = t + C \;\;, \quad C = -t_s - \frac{1}{x(t_s)} \; .$$

This is a way to understand the relation between the "differential equations" (equations involving differentials) (5) (6) and the integrated form (7).

Modeling with differentials

People often use informal reasoning with differentials to create differential equation models of physical systems. For example, suppose $N(t)$ is the number of bacteria in a dish. In a small interval of time dt there the number of bacteria changes by dN. The simple growth model is that dN is proportional to N and dt. The number of new bacteria is proportional to the number of bacteria already there and proportional to the increment of time. We write g for the constant of proportionality (g is for "growth rate"). The model is formulated as

$$dN = gN\,dt \; . \tag{11}$$

[3]The integral sign $\int$ is a big "S", which means "sum". The big Greek letter "sigma" is Σ and also means "sum".

This is truly a “differential equation”, an equation involving differentials. It may be put in textbook form by dividing both sides by dt to get

$$\frac{dN}{dt} = gN \; . \tag{12}$$

Little of this reasoning is exact in the mathematical sense. For one thing, bacterial are not mathematical objects. The simple growth model ignores possible effects such as (maybe) bacteria needing to be a certain age before they divide. Also the number of bacteria is an integer, which means that dN cannot be less than one without being zero. The idea must be that N is so large that $dN = 10$ (say) is small enough relative to N for the differential relation (11) to be about right. The differential equation model is only an approximate description of the real world.

The solution to the model equation, in either of the equivalent forms (11) or (12), is

$$N(t) = N_0 e^{gt} \; . \tag{13}$$

Here N_0 is the value of N when $t = 0$, which we call “initial condition”. This formula may describe results you could get in a biology lab, but it would not describe them exactly. One obvious reason is that the model assumes N values are so large that the exact N, which is an integer, would be impossible to measure exactly. If you had a very small experiment with N values as low as (say) 100, then the differential equation model (11) would be a crude approximation to the actual mechanism that makes N increase over time.