

Final exam practice

About the final exam

Correction, question 2 of full answer, Bessel equation fixed

- You may bring one “cheat sheet”. This is one piece of paper you may refer to during the quiz where you can write anything you want. No other materials such as notes, books, computer, phone, networked watch, etc. are allowed. They must be put away and inaccessible during the quiz.
- The exam will cover **the whole course**, including material tested in the quiz and the midterm. Use the practice quiz questions for examples of midterm questions that test the earlier material. The questions below are only for material covered after the quiz, but the midterm will cover both.
- Write all answers in the answer book (“blue book”) you will be given. The question sheet will not be handed in or graded.
- Points will be deducted for anything you write that is wrong even if you also give a correct answer. Cross out anything you think is wrong.
- Guessing is discouraged. You will get more points for not answering a question than for giving a wrong answer.
- The final will be 110 minutes in recitation on May 7.
- Explain your answers with a few words. True/false answers without explanation may get no points. Calculations without explanations, even if correct, may not get full credit. For True/False questions: if it’s true, say why in a few words (not a formal proof). A mathematical statement is false if there’s a counterexample. For example, “All prime numbers are odd” is false because 2 is a counterexample. If the statement is “All prime numbers are odd”, you can answer: “no, 2 is prime and even”. You probably would get full credit for saying: ”no, 2”
- There will be True/False questions that should be quick to answer and full answer questions that involve reasoning or calculations.
- The practice questions below are only about course content after the midterm. Use practice and actual questions from the midterm and quiz for practice on the rest of the course content.

True/False

1. Consider using the forward Euler method to approximate the trajectory of $\dot{x} = f(x)$ starting with $x_0 = x(0)$ given. We get no information about the solution beyond the time T where $x^h(t)$ is not close to $x(t)$. Here, x^h is the approximate solution produced by the forward Euler method.
2. The work needed to compute an ODE trajectory up to a fixed time T using Euler time-stepping is inversely proportional to h (the time step size).
3. Higher order methods such as the midpoint rule can compute a trajectory up to a given time to a given accuracy with fewer time steps than the first order forward Euler method.

4. Consider trajectories that satisfy $\dot{x} = f(x)$. The sensitivity of the solution $x(T)$ to perturbations in the initial condition $x(0)$ can grow exponentially in T even if the trajectory itself stays bounded.
5. For a linear ODE system $\dot{x} = Ax$ with “initial” date $x(0) = x_0$ there is an eigenvector/eigenvalue solution defined for both $t > 0$ and $t < 0$.
6. For the linear heat equation $u_t = u_{xx}$ with “initial” date $u(x, 0) = u_0(x)$ there is an eigenfunction/eigenvalue solution defined for both $t > 0$ and $t < 0$.

Full answer questions

1. Consider the initial value problem $\dot{x} = t + x^2$ with $x(0) = 1$. The solution may be expressed as a power series

$$x(t) = \sum_{n=0}^{\infty} a_n t^n .$$

Find the terms a_0, \dots, a_6 . *Hint.* One way to do this is to substitute the series into the ODE as an ansatz and find the terms one by one. Another way is to use the ODE to evaluate the derivatives one after another the derivatives

$$\left(\frac{d}{dt}\right)^n x(0) .$$

This can be done by differentiating the ODE with respect to t repeatedly.

2. The Bessel function $J_2(r)$ satisfies the ODE

$$\partial_r^2 J_2 + \frac{1}{r} \partial_r J_2 - \frac{4}{r^2} J_2 = -J_2 .$$

Suppose $J_2(r)$ has a power series representation of the form

$$J_2(r) = \sum_{n=0}^{\infty} a_n r^n .$$

What is the first n where a_n is allowed to be non-zero? Find a recurrence relation that determines a_n from a_{n-1} , a_{n-2} , and possibly a_{n-3} .

3. Let $u(x, t)$ be the temperature representing heat flow in a one dimensional heat conducting rod that goes from $x = 0$ to $x = L$. Suppose the temperature is specified to be zero at $x = 0$ and the heat flux is set to be zero at $x = L$.

- (a) Write the diffusion equation PDE and boundary conditions for this situation.
- (b) Find the mode that decays to zero most slowly. A *mode* is a solution of the form $u(x, t) = e^{rt}v(x)$.
- (c) Is it true that modes v_1 and v_2 with eigenvalue $r_1 \neq r_2$ are necessarily orthogonal?

4. Consider the ODE

$$\ddot{x} = -x - \gamma \dot{x} + f(t) .$$

Suppose $f(t)$ is periodic with period 2π and represented by the Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int} .$$

- (a) Assume that $\gamma \neq 0$. Show that there is a periodic solution $x(t)$ represented by a Fourier series

$$x(t) = x_p(t) = \sum_{n=-\infty}^{\infty} b_n e^{int} .$$

Find a formula for b_n in terms of x_n .

- (b) Assume that $\gamma > 0$. Let $x(t)$ be the solution to the ODE with $x(0) = 0$. Show that $x(t) \rightarrow x_p(t)$ as $t \rightarrow \infty$ in the sense that $|x(t) - x_p(t)| \rightarrow 0$ as $t \rightarrow \infty$.
- (c) Assume that $\gamma < 0$ (a non-physical negative “friction” coefficient). Is it true that $x(t) \rightarrow x_p(t)$ as $t \rightarrow \infty$?
5. Write a Fourier sine series solution formula for the solution of the heat equation $u_t = u_{xx}$ with Dirichlet boundary conditions $u = 0$ when $x = 0$ and $x = 1$. The solution is defined for $t > 0$ and $0 \leq x \leq 1$. The initial data are $u(x, 0) = \cos(\pi x)$. *Hint.* (for the algebra)

$$\begin{aligned} \sin(\theta) \cos(\theta) &= \frac{e^{i\theta} - e^{-i\theta}}{2i} \frac{e^{i\theta} + e^{-i\theta}}{2} \\ &= \frac{e^{2i\theta} - e^{-2i\theta}}{4i} \\ &= \frac{1}{2} \frac{e^{2i\theta} - e^{-2i\theta}}{2i} \\ &= \frac{1}{2} \sin(2\theta) \end{aligned}$$