75 minute Midterm exam

About the quiz

- You may bring one "cheat sheet". This is one piece of paper you may refer to during the quiz where you can write anything you want. No other materials such as notes, books, computer, phone, networked watch, etc. are allowed. They must be put away and inaccessible during the quiz.
- Write all answers in the answer book ("blue book") you will be given. The question sheet will not be handed in or graded.
- Points will be deducted for anything you write that is wrong even if you also give a correct answer. Cross out anything you think is wrong.
- Guessing is discouraged. You will get more points for not answering a question than for giving a wrong answer.
- Explain your answers with a few words. True/false answers without explanation may get no points. Calculations without explanations, even if correct, may not get full credit. For True/False questions: if it's true, say why in a few words (not a formal proof). A mathematical statement is false if there's a counterexample. For example, "All prime numbers are odd" is false because 2 is a counterexample. If the statement is "All prime numbers are odd", you can answer: "no, 2 is prime and even". You probably would get full credit for saying: "no, 2"

True/False

1. If a linear system $\dot{x} = Ax$ has a matrix A with all negative eigenvalues (strongly stable) then the norm of the solution is decreasing in time: $\frac{d}{dt}\|x(t)\| \leq 0$. Use the euclidian norm $\|x\| = \left(x^T x\right)^{\frac{1}{2}}$. *Hint.* The matrix $A = \begin{pmatrix} -5 & 12 \\ -1 & 2 \end{pmatrix}$ has $\lambda_1 = -1$ and $\lambda_2 = -2$.
2. If $\dot{x} = f(x)$ is not globally lipschitz then it has a solution that blows up in finite time.
3. If $\dot{x} = f(x)$ (two components) has a critical point at x_* that is a saddle, then there are solutions $x(t)$ that approach x_* as $t \to \infty$ and solutions. Give an answer that assumes the linearized equation correctly predicts the non-linear behavior. Do not give a proof.
4. Suppose $x(t)$ is an n component vector and satisfies a linear ODE system $\dot{x} = Ax$. The set of initial data $x(0) = x_0$ with $x_1(3) = 0$ is a linear subspace of $\mathbb{R}^n$. Here, $x_1(t)$ is the first component of $x(t)$.

Full answer questions

1. Consider the initial value problem

$$\dot{x} = \tfrac{1}{3}x^3 \quad , \quad x(0) = x_0 \ .$$

   (a) Find a formula for $x(t)$ when $x_0 = 1$.

   (b) Find a formula for the blow-up time T_* as a function of x_0.

(c) (*harder*) Show that x and t may be rescaled (that is, $x(s) = ay(\tau s)$) so that $\frac{dy}{ds} = \frac{1}{3}y^3$ and $y(0) = 1$. Show that the answer to part (b) is a consequence of this and part (a).

2. Find the solution of the initial value problem $\dot{x} = Ax + c$, specifically,

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -6 & 2 \\ -13 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \quad x_0 = x(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

You will get partial credit for each of the following steps:

- Find eigenvalues and eigenvectors
- Find the steady state $x_* = \begin{pmatrix} a \\ b \end{pmatrix}$ with $0 = Ax_* + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- Find initial values y_0 so that $x(t) = y(t) + x_*$, with $\dot{y} = Ay$, satisfies the initial value problem
- Express y_0 as a linear combination of eigenvectors of A
- Assemble these pieces to give an expression for $x(t)$

3. Consider the nonlinear differential equation system

$$\begin{aligned} \dot{x_1} &= -x_2 + x\sin(x_1x_2) \\ \dot{x_2} &= -2x_1 - x_2\sin(x_1x_2) \end{aligned}$$

(a) Find the critical point (fixed point). More precisely, show that there is one and only one fixed point and find it. Call this point $x_* = \begin{pmatrix} x_{1*} \\ x_{2*} \end{pmatrix}$. *Hint.* Define $a = \sin(x_1x_2)$. For a given a, you get a linear system for x_1 and x_2. Is there an allowed a value for which the solution is not unique?

(b) Write these in the form of a two component nonlinear ODE $\dot{x} = f(x)$ and find the jacobian matrix $f'(x_*)$.

(c) Find eigenvalues and eigenvectors of $f'(x_*)$ and use them to determine the behavior of trajectories near the critical point.

(d) Make a phase plane drawing that shows this behavior near the critical point. Eigenvalues and eigenvectors are both relevant for the drawing.

4. Suppose $z(t) = (1+i)e^{3it}$ and $x(t) = \text{Re}(z(t)$. Find a formula of the form $x(t) = A\cos(\omega(t-t_0))$. Find formulas for A, ω, and t_0.