

Midterm practice

About the midterm exam

- You may bring one “cheat sheet”. This is one piece of paper you may refer to during the quiz where you can write anything you want. No other materials such as notes, books, computer, phone, networked watch, etc. are allowed. They must be put away and inaccessible during the quiz.
- The midterm will cover everything we have covered up to now, including material tested in the quiz. Use the practice quiz questions for examples of midterm questions that test the earlier material. The questions below are only for material covered after the quiz, but the midterm will cover both.
- Write all answers in the answer book (“blue book”) you will be given. The question sheet will not be handed in or graded.
- Points will be deducted for anything you write that is wrong even if you also give a correct answer. Cross out anything you think is wrong.
- Guessing is discouraged. You will get more points for not answering a question than for giving a wrong answer.
- The midterm will be 75 minutes in recitation on March 27.
- Explain your answers with a few words. True/false answers without explanation may get no points. Calculations without explanations, even if correct, may not get full credit. For True/False questions: if it’s true, say why in a few words (not a formal proof). A mathematical statement is false if there’s a counterexample. For example, “All prime numbers are odd” is false because 2 is a counterexample. If the statement is “All prime numbers are odd”, you can answer: “no, 2 is prime and even”. You probably would get full credit for saying: ”no, 2”
- There will be 4 True/False questions that should be quick to answer and 4 full answer questions that involve reasoning or calculations.
- The practice questions below are to let you know the kind of questions that might be asked. The actual midterm will may be different, but in a similar spirit.

True/False

1. The ODE $\dot{x} = -(1 + t^2)x^3 + \sin(t)$ has solutions that blow up in finite time. *Hint.* Do not look for a formula for the solution, there isn’t one.
2. Consider a linear ODE system $\dot{x} = Ax$ with initial data $x(0) = x_0$, The initial point x_0 is called *stable* (more properly, *asymptotically stable*, if $x(t) \rightarrow 0$ as $t \rightarrow \infty$. The set of stable initial points is a linear subspace.
3. Consider a linear ODE system $\dot{x} = Ax$ with initial data $x(0) = x_0$, The initial point x_0 is called *unstable* if $x(t) \rightarrow 0$ as $t \rightarrow \infty$. The set of unstable initial points is a linear subspace. *Hint.* Find the stable and unstable initial points for the ODE system with $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

4. It is possible that a fixed point for a nonlinear ODE system $\dot{x} = f(x)$ to be neither linearly stable nor linearly unstable.
5. A nonlinear ODE system can have at most one linearly stable fixed point.
6. An ODE, linear or nonlinear, always has at least one fixed point.
7. A linear n component ODE system has an *invariant line* if there is an initial condition $x_0 \neq 0$ so that the solution has $x(t) = m(t)x_0$. [All points in the trajectory $x(t)$ are on the same line in \mathbb{R}^n .] Here, $x(t)$ is a vector and $m(t)$ is a *scaling factor*. Every two dimensional linear system has an invariant line.
8. A linear n component ODE system has an *invariant line* if there is an initial condition $x_0 \neq 0$ so that the solution has $x(t) = m(t)x_0$. [All points in the trajectory $x(t)$ are on the same line in \mathbb{R}^n .] Here, $x(t)$ is a vector and $m(t)$ is a *scaling factor*. Every three dimensional linear system has an invariant line.
9. If $\dot{x} = f(x)$ has $f(x_*) = 0$ and the eigenvalues of $f'(x_*)$ all have negative real part, then if the starting point $x(0) = x_0$ is close enough to x_* then $x(t) \rightarrow x_*$ as $t \rightarrow \infty$.
10. If a nonlinear ODE system has a positive Liapunov exponent then the solution at large time may be essentially unpredictable even though the motion is not random.
11. If a nonlinear ODE model has a positive Liapunov exponent then the model is useless for predicting large time properties of solutions because the solution at a large time depends so sensitively on the initial condition.

Full answer questions

1. Suppose $x_1(t)$ and $x_2(t)$ are two sinusoidal signals with the same frequency, ω , and amplitude, A , but with a phase difference of $\frac{\pi}{3}$. Find the amplitude of the sum $y(t) = x_1(t) + x_2(t)$. *Hint.* You may take $x_1(t) = A \operatorname{Re}(e^{it})$ and $x_2(t) = A \operatorname{Re}(e^{it+i\frac{\pi}{3}})$ (implicitly taking $\omega = 1$, why?), Then $e^{i\frac{\pi}{3}} = r + is$ and $1 + r + is = me^{i\theta}$ with $m^2 = \sqrt{(r+1)^2 + s^2}$ (why?).
2. Consider the two component linear system

$$\dot{x} = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix} x .$$

- (a) Find an expression for its fundamental solution matrix $S(t)$.
- (b) Show by explicit matrix multiplication that

$$S(t_1)S(t_2) = S(t_1 + t_2) .$$

- (c) Find a formula for $x_2(0)$ so that $x_1(0) = 1$ and $x_1(3) = 0$.

3. Find a basis for the vector space of solutions of the ODE

$$\frac{d^3x}{dt^3} + \frac{dx}{dt} + 10 = 0 .$$

Hint. If you find one root of a cubic $p(r) = 0$, you can “deflate” to find a quadratic satisfied by the other roots. That is, if $p(r_*) = 0$, then you can find $q(r) = p(r)/(r - r_*)$, using “long division” of polynomials. Look for r_* by trying integers.

4. Consider the ODE

$$\ddot{y} = -y^3 + y - \dot{y} .$$

- (a) Formulate this as a system of two first order ODEs in two variables $x_1 = y$, $x_2 = \dot{y}$.
- (b) Find the stationary points (critical points).
- (c) Do a linearized analysis for each stationary point, including
- Find the linearized ODE
 - Find the eigenvalues and eigenvectors of the corresponding matrix.
 - Draw a phase plane diagram and use the eigenvalues and eigenvectors to illustrate behavior of trajectories near the critical points.
- (d) Determine (qualitatively, based on computing and plotting values of the vector field at enough points and along the axes) the behavior of trajectories for larger x .