Probability Limit Theorems, II, Homework 2

(1) Suppose that $d\mu(x)$ is a probability measure with

$$\int x^n d\mu(x) = \frac{1}{\sqrt{2\pi}} \int x^n e^{-x^2/2} dx$$

for all positive n. Show then that $d\mu(x) = \frac{1}{\sqrt{2\pi}} \int x^n e^{-x^2/2} dx$. Hint: Applying Tchebychev's inequality to appropriate moments yields

$$\Pr\left(|X| \geq R\right) \leq Ce^{-R^p} \quad \text{for some } p > 1.$$

This implies that the Fourier transform (characteristic function)

$$\hat{\mu}(\zeta) = \mathrm{E}_\mu\left[e^{i\zeta X}\right]$$

is an entire analytic function of $\zeta = \xi + i\eta$. The moment conditions then imply that

$$\hat{\mu}(\xi) = e^{-\xi^2/2} \quad .$$

Now, for any interval, (a, b), express $\mu((a, b))$ as an integral involving $\hat{\mu}(\xi)$.

(2) Let D be a bounded open set and $q \in \partial D$. For any $x \in D$ and $R > |x - q|$, define

$$\tau = \inf\left\{t \mid W(t) + x \in \partial D \text{ or } |W(t) + x - q| \geq R\right\} \ .$$

Define $h(x, R) = \Pr\left(W(\tau) + x \in \partial D\right)$, and $p(\alpha, R) = \inf_{|x-q|=\alpha} h(x, R)$, and $l(R) = \liminf_{\alpha \to 0} p(\alpha, R)$. Show that if $l(R) > 0$ then $l(R) = 1$.

(3) As in problem (2), let E be the complement of $\overline{D}$, the closure of D. For any set A, $|A|$ will be the Lebesgue measure of A. For $q \in \partial D$. let

$$v(q) = \limsup_{R \to 0} \frac{|E \cap B_R(q)|}{|B_R(q)|} \quad .$$

Use problem (2) to help show that if $v(q) > 0$ then q is a regular point. In particular, show that if q satisfies the exterior cone condition then q is a regular point. Remember that if $W(t) + x \in E$, then $\tau < t$. Try to show that $\Pr\left(W(t) + x \in E\right) > 0$.