Probability Limit Theorems, II, Homework 3,
Mehler formula and Hermite polynomials

The computations here revolve around the "velocity part" of the Ornstein Uhlenbeck process

$$dX = -X dt + dW \; . \qquad (1)$$

The normalization does not make physical sense but simplifies later computations. The PDE satisfied by the probability density for $X(t)$ is

$$u_t = \frac{1}{2} u_{xx} + (xu)_x = u_{xx} + x u_x + u = Lu \; . \qquad (2)$$

The backward equation, satisfied by expected values, is

$$f_t + f_{xx} - x f_x = f_t + L^* f = 0 \; . \qquad (3)$$

The operators $Lu = \frac{1}{2} u_{xx} + (xu)_x$ and $L^* f = \frac{1}{2} f_{xx} - x f_x$ are adjoint in the sense that

$$\langle f, Lu \rangle = \langle L^* f, u \rangle \;\; ,$$

with $\langle \cdot, \cdot \rangle$ being the L^2 inner product.

(1) We begin with a PDE approach to finding the fundamental solution. This is the function, $G(x,y,t)$, that satisfies (2) as a function of x and t, together with initial conditions $G(x,y,0) = \delta(x-y)$. In view of the formula

$$\delta(x-y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi(x-y)} d\xi$$

we may write

$$G(x,y,t) = \frac{1}{2\pi} \int e^{-i\xi y} u(x,t,\xi) d\xi \; , \qquad (4)$$

where u satisfies (2) with initial data $u(x,t,\xi) = e^{i\xi x}$. This approach can work because the geometric optics construction of plane wave solutions is exact in this case. That is, (2) has exact solutions of the form

$$u(x,t) = A(t) e^{i\xi(t)\cdot x} \; .$$

Find these solutions, and you will find that the integral (4) can be found in closed form. Use this method to find a closed form expression for $G(x,y,t)$, which is the Mehler formula.

2. For a probabilist, a simpler approach may be to compute the probability density for $X(t)$ directly from (1). The solution of (1), with initial data $X(0) = y$ (corresponding to $G(x,y,0) = \delta(x-y)$) is (check this)

$$X(t) = e^{-t} y + \int_0^t e^{-(t-s)} dW(s) \; .$$

From this it is obvious that $X(t)$ is Gaussian. The mean and variance are easy to compute. Use this to get the density for $X(t)$. Check that this agrees with your answer to question 1.

3. We express the solution of (2) as

$$u(x,t) = \sum_n a_n e^{\lambda_n t} \phi_n(x) \ \ ,$$

where the ϕ_n and λ_n are the eigenfunctions and eigenvalues of the operator L:

$$L\phi_n = \frac{1}{2}\phi_{nxx} + x\phi_{nx} + \phi_n = \lambda_n \phi_n \ . \tag{5}$$

We find the coefficients, a_n, in the following way. A general function, $g(x)$ can be written

$$g(x) = \sum_n \widehat{g}_n \phi_n(x) \ ,$$

where

$$\widehat{g}_n = \int \psi_n(x) g(x) dx = \langle \psi_n, g \rangle \ \ ,$$

and the ψ_n are the "adjoint eigenfunctions", which satisfy

$$L^*\psi_n = \frac{1}{2}\psi_{nxx} - x\psi_{nx} = \lambda_n \psi_n \ ,$$

subject to the normalization

$$\langle \psi_n, \phi_n \rangle = \delta_{mn} \ .$$

We can find the ϕ_n by converting the eigenvalue problem (5) into the harmonic oscillator eigenvalue problem. Write $\phi_n(x) = w(x)h_n(x)$, with $w'/w + x = 0$, and you get

$$-\mathcal{H}h_n = \left(\lambda_n - \frac{1}{2}\right) h_n \ \ , \quad \text{where} \quad \mathcal{H}g = \frac{-1}{2} g_{xx} + \frac{1}{2}x^2 g \ . \tag{6}$$

There is a similar trick for the adjoint eigenfunctions. Use this to write a formula for the Green's function kernel in terms of the ϕ_n and ψ_n, that is, in terms of Hermite polynomials.

4. Suppose we solve (2) with "general" initial data, $u(x,0) = \rho(x)$, that is a probability density. This is the same as Taking initial data $X(0)$ for (1) from the density ρ. Use the results of part 3 to show that $u(\cdot,t)$ converges to the standard normal density exponentially fast with a rate that depends on the number of Hermite polynomials that are orthogonal to ρ.