Probability Limit Theorems, II, Homework 4, Hermite polynomials and Wiener chaos

Probabilists often use a different normalization for Hermite polynomials than the one in homework 3. Adopting this, we define $H_n(x)$ by

$$H_n(x)e^{-x^2/2} = \partial_x e^{-x^2/2} \ . \tag{1}$$

It is common to put in a factor of $(-1)^n$ so that H_n has the form $x^n + \cdots$. Our H_3 is $-x^3 + 3x$, not $x^3 - 3x$.

1. The exponential generating function (as opposed to the "ordinary" generating function) for the H_n is

$$F(x,z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x) \ .$$

The ordinary generating function is missing the factor $1/n!$. Use the formula (1) to find $F(x,z)$ explicitly. Use this to find an integral formula for $H_n(x)$. This formula can be used to derive approximations to $H_n(x)$ when n and x are large.

2. With the notation $g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, show that

$$\int H_n(x) H_m(x) g(x) dx = 0 \qquad \text{when } m \neq n,$$

and evaluate

$$\int H_n(x)^2 g(x) dx \ .$$

Hint: Show that $x^k e^{-x^2/2} \sim H_k(\partial_x) e^{-x^2/2}$ (e.g. using the Fourier transform). Suppose that $m < n$ and show that

$$H_m(x) \partial_x^n e^{-x^2/2} = \partial_x^{n-m} P(\partial_x) e^{-x^2/2} \ ,$$

where P is some polynomial. For $m = n$ this is almost true.

3. Show that the Hermite polynomials are a (complete) basis for the Hilbert space L^2_g, which has inner product

$$\langle u, v \rangle_g = \int \overline{u}(x) v(x) g(x) dx \ .$$

Do do this, compute the Hermite polynomial expansion of $e^{i\xi x}$ (formula (1) will be helpful here) and show explicitly that it converges to $e^{i\xi x}$. Why is this enough?

4. Suppose that X is a standard normal random variable and $f(x)$ has $E(f^2(X)) < \infty$. Interpret the Hermite expansion of f as a representation of the random variable $f(X)$ as a linear combination of "Hermite" random variables $H_n(X)$.

5. Here is the extension to n variables. A multi-index is a list of n non negative integers: $\alpha = (\alpha_n, \cdots, \alpha_n)$. The notation x^α means $x_1^{\alpha_1} \cdot \cdots \cdot x_n^{\alpha_n}$, and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdot \cdots \cdot \partial_{x_n}^{\alpha_n}$. The degree of a multi index is $p = |\alpha| = \alpha_1 + \cdots + \alpha_n$. The multidimensional Hermite polynomials are

$$H_\alpha(x) = H_{\alpha_1}(x_1) \cdot \cdots \cdot H_{\alpha_n}(x_n) = e^{|x|^2/2} \partial_x^\alpha e^{-|x|^2/2} \ . \tag{2}$$

Show that the Hilbert space $L_g^2(R^n)$ is a direct orthogonal sum of the degree p subspaces

$$\mathcal{S}_p = \text{span of } \{H_\alpha \ \mid \ |\alpha| = p\} \ .$$

6. For any orthogonal $n \times n$ matrix, Q, and any function $f \in L_g^2(R^n)$, define Qf by $(Qf)(x) = f(Qx)$. Clearly $\|Qf\|_g = \|f\|_g$.

a. Verify by direct calculation that the space $\mathcal{S}_2$ is invariant under the action of Q for $n = 2$. That is, show that if $x' = x\cos(\theta) + y\sin(\theta)$ and $y' = y\cos(\theta) - x\sin(\theta)$, then

$$H_2(x') = a(\theta)H_2(x) + b(\theta)H_1(x)H_1(y) + c(\theta)H_2(y) \ ,$$

and find a similar representation for $H_1(x')H_1(y')$.

b. Give a simpler proof that each $\mathcal{S}_p$ is invariant for any n. Hint: what does Q do to ∂_x^α?

c. Compute $f_p(x, \xi) = \mathcal{S}_p e^{i\xi\cdot x}$. Here, I have used $\mathcal{S}$ to denote the orthogonal projection onto the space $\mathcal{S}$. Hint: the right Q can make this easy.

7. Now let $(\Omega, \mathcal{F}, \mu)$ represent Wiener measure for Brownian motion on the interval $[0, 1]$ with $W(0) = 0$. Let $\mathcal{F}_L \subset \mathcal{F}$ be the algebra of sets generated by the diadic interval differences

$$G_{L,k} = W((k+1)2^{-L}) - W(k2^{-L}) \ , \quad \text{for } k = 0, 1, \ldots, 2^L - 1.$$

For any $f(W)$ (I will use W instead of ω to represent the basic random element of Ω: an element of Ω is a path.) with $E[f^2(W)] < \infty]$ define

$$f_{L,p}(W) = \mathcal{S}_p \mathrm{E}\left[f \mid \mathcal{F}_L\right] \ .$$

Here, we use $\mathcal{S}_p$, as in part 6c, to be the orthogonal projection onto the space of order p in the space of functions of $n = 2^L$ independent gaussians. Show that, for each p, $f_{L,p}$ is a martingale as a function of L. Show that the limit

$$\lim_{L\to\infty} f_{L,p}(W) = f_p(W)$$

exists, that $f(W) = \sum_{p=0}^{\infty} f_p(W)$, and that

$$\mathrm{E}\left[f^2(W)\right] = \sum_{p=0}^{\infty} \mathrm{E}\left[f_p^2(W)\right] .$$

8. Show that for each f and p there is a function $a_p(t_1, \ldots, t_p)$ with

$$f_p(W) = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{p-1}} a_p(t_1, \ldots, t_p) dW(t_p) \cdots dW(t_1) .$$

Hint: Take a limit of approximations $a_{L,p}(t_1, \ldots, t_p)$, which are martingales (in L) for each p as functions of $(t_1, \ldots, t_p)$ (show this). Note that in the limit, the energy in all terms involving $H_2(x) = x^2 - 1$ or higher goes to zero. These spaces are the Wiener chaos spaces of degree p.