Probability Limit Theorems, II
Final Assignment

1. Suppose X has a continuous bounded density, $f(x)$, finite fourth moment, and $E(X) = 0$. Define $S_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k$, where the X_k are independent samples from f. Let $f_n(x, s)$ be the density (as a function of x) for $E[X_1 | S_n = s]$.

a. Show that $f_n(x, 0) \to f(x)$ as $n \to \infty$. Hint: calculate $E[e^{i\xi X_1 + i\eta S_n}]$.

b. For fixed a, show that, under reasonable hypotheses,

$$f_n(x, \sqrt{n} \cdot a) \to \frac{1}{Z(\mu_*)} f(x) e^{\mu_* x}$$

for suitable choice of $\mu_*(a)$.

2. Let μ be a nonatomic probability measure on the space of continuous functions of one variable that are periodic with period 2π. For any Borel set of continuous periodic functions (in the sup norm), A, let A_k be the set of functions $e^{ikx} f(x)$ for $f \in A$. Prove that $\mu(A_k) \to 0$ as $k \to \infty$. Hint:

a. Show that in a Polish space (complete metric space with a countable dense set), a set A is compact if and only if it is "totally bounded". A set is totally bounded if, for any $\epsilon > 0$, there are finitely many balls of radius epsilon that cover A.

b. Let μ be a probability measure on a Polish space, show that for any $\epsilon > 0$ and $\alpha < 1$ there is a finite collection of balls of radius $\leq \epsilon$ so that $\mu(\bigcup_k B_k) > \alpha$. From this, show that there is a totally bounded set, A, with $\mu(A) > \alpha$.

c. Show that a closed set of periodic functions is compact if and only if they have a common modulus of continuity.

3. The hypercube graph has as vertices the vertices of the unit hypercube in d dimensions. A vertex is described by a sequence of bits; $v = (s_1, \ldots, s_d)$, with $s_k \in \{0, 1\}$. The "partiy" of a vertex is the sum of the bits, mod 2: even if $\sum_k s_k$ is even and odd otherwise. A random walk on the hypercube graph chooses a component, $k \in \{1, \ldots, d\}$ at random with each k equally likely and reverses the bit s_k ($0 \to 1$ and $1 \to 0$).

a. Show that this Markov chain is not ergodic but that the two step chain (the map $X(t) \to X(t+2)$) is ergodic on the even or odd parity vertices.

b. What is the invariant measure, μ, for this chain on the even parity vertices?

c. Construct a coupling argument to show that

$$\sum_{v \in \mathcal{E}} |Pr[X(2t) = v] - \mu(v)| \leq \exp(-\alpha t / d \log(d)) \ ,$$

for a positive constant, α, that is independent of d or the probability distribution of $X(0)$. Hint: make sure that the number of bits of agreement between the two walkers does not decrease but does increase whenever possible.

4. Let $C(x)$ be a positive "covariance" function on periodic functions. That is, if $f(x)$ and $g(x)$ are periodic functions, then $\langle f, g \rangle = \int f(x)C(x-y)g(y)dx$ has $\langle f, f \rangle > 0$ whenever f is not identically 0. A Gaussian random function with covariance C is a random periodic function, $F(x)$, so that any finite collection of sample values is jointly gaussian and $\text{cov}(F(x), F(y)) = C(x-y)$.

a. Give a formal derivation of the fact that a gaussian random function with covariance C has Fourier coefficients that are independent gaussians and find the formula for their variances.

b. Show that if $|C(x)| \leq \mathit{const}|x|^\alpha$ for some positive α, then there is a probability measure on the continuous periodic functions that is gaussian and has covariance function C.

5. The Cameroon-Martin-Girsanov formula may be used to give a "weak" solution to the stochastic differential equation

$$dX = u(X)dt + dW(t)$$

even when the coefficient $u(x)$ is not lipschits continuous. Let us suppose only that $u(x)$ is bounded and measurable.

a. Show that paths $X(t)$ are Hölder continuous with probability 1 for any Hölder exponent less than $1/2$. Hint: this is *very* easy; don't use the method from question 4.

b. Show from the Cameroon-Martin-Girsanov formula that

$$X(T) - \int_0^T u(X(t))dt - X(0)$$

is a Brownian motion.