Final exam, December 21

(minor typos corrected and small explanations added)

Name:

Instructions:

- Explain your reasoning. Points may be subtracted even for correct answers otherwise.
- Cross out anything you think is wrong. Points may be subtracted for wrong answers even if the correct answer also appears.
- You will receive 20% credit for a blank answer. You may have points deducted from this for a wrong answer.
- Answer each question in the space provided.
- You are allowed one "cheat sheet", which is a standard size piece of paper with whatever you want written on it.

1. Consider continuously compounded interest rate r and a possibly different simple interest payment $1 \longrightarrow 1 + RT$ at time T. The continuously compounded interest on one unit of currency yields $1 \longrightarrow 1 + M_T$ at time T.

   (a) Write a formula for M_T.

   (b) Suppose $r = R = 5\%/\text{year}$ and T is two years. Use Taylor series to the appropriate order to estimate how much more M_T is than RT.

   (c) Suppose $R = 5\%/\text{year}$, T is two years, and $M_T = RT$. Estimate how much less r is than 5%/year. Give the result in basis points (e.g., r is $\cdots$ basis points less than 5%/year.)

2. Suppose there is a market for three "bets", which are financial contracts whose payout depends on the outcome of a race. "Bet 3" is a contract that pays 1 if competitor C comes in first, second, or third. "Bet 2" pays R_2 if C comes in first or second and nothing if C is third. "Bet 1" pays R_1 if C comes in first and nothing if C is second or third. The competitor is very fast, and is guaranteed to come in at least third. All bets may be bought or sold for price 1 today.

   (a) Show that this market is arbitrage free if and only if $1 < R_2 < R_1$.

   (b) Find the risk neutral probability that C comes in first, second, or third in terms of the numbers R_2 and R_3, assuming there is no arbitrage.

3. We have seen that mean/variance analysis can lead to portfolio weights that are large positive and negative numbers. It has been proposed to reduce the sizes (absolute values) of portfolio weights by putting a constraint on

$$S = \sum_{j=1}^{n} w_j^2 \ .$$

   (a) Find the minimum value of S subject to the budget constraint (in our usual notation)

   $$\mathbf{1}^t w = 1 \ .$$

   Call this minimum value s_0.

   (b) Is there a maximum value of S among portfolio weights that satisfy the budget constraint? Why or why not? Remember that portfolio weights are allowed to be negative.

   (c) Suppose C is a symmetric and positive definite covariance matrix for the returns R_j. Write the Lagrange multiplier equation (in matrix/vector form) for the minimum variance portfolio subject to the budget constraint and the constraint $S = s_1 > s_0$.

   (d) Explain an algorithm using linear algebra (solving systems of linear equations, inverting matrices, etc.) and bisection search that can find the values of the two Lagrange multipliers.

4. Suppose that X_0, X_1, …, X_n are independent random variables with

$$a_k = \mathrm{E}[\,X_k\,] \quad , \qquad s_k^2 = \mathrm{var}(\,X_k\,) \ .$$

We call X_0 the *market factor*, and X_1, …, X_n are *idiosyncratic* factors. The asset returns are

$$R_k = X_k + \beta_k X_0 \quad , \qquad \text{for } k = 1, \ldots, n.$$

(a) In this model, find formulas for the expected returns $\mu_k = \mathrm{E}[\,R_k]$, and the covariances $C_{jk} = \mathrm{cov}(\,R_j, R_k)$.

(b) Assume that $s_k^2 > 0$ for each k. Show that the covariance matrix C is positive definite. Hint: it is not enough to show that the diagonal entries C_{kk} are positive.

5. Let S_t be the price of a stock at time t. Suppose the stock price has volatility σ and expected rate of return μ. The risk free rate of return is r. Consider an American style put option with strike K and expiration time T.

   (a) Explain the binomial tree algorithm for finding the put price. Assume the binomial tree has times $t_k = k\Delta t$ so that $t_n = T$. Give the relevant formulas for propagating the price backward through the tree.

   (b) Suppose the options pays a continuous coupon c until it is exercised. As long as you hold the option, you receive a payment $c\Delta t$ in a time interval of length Δt. The payments stop once you exercise the option, or when the option expires. Explain how the pricing algorithm would change because of the coupon payments. Give any relevant formulas. Hint: in a one period binary model, what effect does the coupon payment have? You may approximate the continuous payments to be up to n discrete payments of size $c\Delta t$ at times t_k.

6. A European style *digital option* pays one unit of currency if $S_T < K$ and nothing if $S_T \geq K$. Find the Black Scholes price for a digital option as a function of σ (the stock volatility), r (the risk free rate), S_0 (the spot price), and K. Hint: This uses the Black Scholes theory (log-normal stock price) but not the Black Scholes formula. Express S_T in terms of the exponential of a Gaussian (identify the mean and variance) or in terms of a standard normal. Then write the event that the option is in-the-money (has a non-zero payout) in terms of this Gaussian. Then write the the probability of a payout in terms of the cumulative normal (the N function). Remember to discount.

7. Suppose `S[k]` is the price of a stock at the close of trading k trading days in the past. Suppose that we have n closing prices. Write code in `R` that estimates the expected return and the variance of the return over this period. Put the results into variables `mu` (for μ) and `sigsq` (for σ^2) respectively. Hint: the first step is to calculate the daily returns from the closing prices.