Sample questions for the final exam, Monday, December 21, **8 am**. Also review the quiz, midterm, and their sample questions

Instructions (for the midterm):

- Explain your reasoning. Points may be subtracted even for correct answers otherwise.
- Cross out anything you think is wrong. Points may be subtracted for wrong answers even if the correct answer also appears.
- You will receive 20% credit for a blank answer. You may have points deducted from this for a wrong answer.
- Answer each question in the space provided.
- You are allowed one "cheat sheet", which is a standard size piece of paper with whatever you want written on it.

Questions (the midterm will be shorter than this):

1. Suppose that the risk free yield on a 1 year bond is .25%, which is 25 basis points. Suppose the yield on a 2 year bond is .6%. What is the forward yield for a one year bond starting a year from today? Here's the abstract version of the same question: Suppose r_1 is the yield from today to time t_1 and r_2 is the yield up to time t_2. In terms of these numbers, how would you replicate a loan given at time t_1 and paid at time $t_2 > t_1$? Assuming that $r_1 t_1$ and $r_2 t_2$ are small, use Taylor series (first or second derivative approximations) to give a simplified version of the answer. The point of this question is to redo the forward price for bonds stuff, but with yields instead of the quantities $P(t_1, t_2)$ and F.

2. The *secant method* is a possibly better (faster) way to solve a nonlinear equation. Suppose you want to find x with $f(x) = 0$. Suppose you have a with $f(a) < 0$ and b with $f(b) > 0$. The secant method constructs the line (the equation for the line) that passes through these points:

$$l(x) = f(a) + (x-a)\frac{f(b) - f(a)}{b-a} \; .$$

The algorithm then finds c with $l(c) = 0$. It evaluates $f(c)$ and reduces the interval of uncertainty either to $[a, c]$ or to $[c, b]$.

   (a) Draw a diagram that illustrates the possibility that c may be a better approximation to the root x than the midpoint guess $\frac{1}{2}(b-a)$.

(b) Write code in R that is similar to the bisection algorithm code you used for assignment 9 but does the secant method instead. Just change what needs changing the function that does the search.

3. **Risk neutral pricing, the abstract version**. Suppose there are n assets in the "market". Each asset can be bought or sold in whatever quantity at the market price. Today (time $t = 0$, each asset price is 1 (or \$1, though the currency is irrelevant). *Tomorrow*, at time T, there are n possible "states of the market". In state j, the price of asset i is P_{ij}. The numbers P_{ij} are known today, but the value of j is not known today. The "real world" probability of j is not relevant for this problem, except that each real world probability is positive so that each state, j, is possible. The prices P_{ij} are the entries of an $n \times n$ matrix, P.

   Now consider an "option" that makes payout Q_j in state j at time T. We want to *replicate* Q with a portfolio of market assets. The replicating portfolio is characterized by the *portfolio weights* which are called w_i. This is the amount of asset i in the replicating portfolio. *Replication* means that for each state of the market tomorrow, j, the value of the portfolio is equal to the payout of the option:

$$Q_j = \sum_{i=1}^{n} w_i P_{ij} \ . \tag{1}$$

   This is a system of n linear equations for the n unknowns w_i. The market is *complete* if every option can be replicated using existing assets. That means that the market is complete if and only if the matrix P is invertible.

   The next step assumes that there is a *risk free asset*, which is an asset whose price tomorrow is known today, and therefore is independent of j, which is the state of the market tomorrow. Note: an asset is said to be *risky* if its value is unknown. This does not imply "risk" as we usually understand the term. It might be that there is an asset i so that P_{ij} is a large positive number for each j. In that case, the only "risk" is the unknown size of your profit. *Risk* is related to the variance of an asset price. In this sense, the unknown prices $-P_ij$ (as a function of the state j for fixed i) are just as risky as P_{ij}. Suppose asset 1 is *risk free.* This means that P_{1j} is the same for each j. We call this number $\frac{1}{B}$ and assume it's positive:

$$P_{1j} = \frac{1}{B} \ , \qquad \text{for all } j \ .$$

   Here "B" is for *bond.* If you pay B today, you get 1 at time T. It's natural to think $B < 1$ (which is the same as a positive rate of return), but this is not necessary for the theory, and is not always true in real markets. In Germany today, the risk free rate (the return on Euro denominated short term German government bonds) is negative.

The "replicating portfolio" theory of pricing says that the market price today of Q should be the market price today of the replicating portfolio:

$$\text{Price}(Q) = \sum_{i=1}^{n} w_i \; .$$

The reason is that if the price of Q were more than the price of the replicating portfolio, then it would be possible to "arb" the option by selling Q, using some of the money to buy the replicating portfolio and the keeping the rest. Since the portfolio exactly replicates the option, at time T, you sell the portfolio and use the proceeds to make the Q payout. The formulas (1) say that this is possible for any state j. If Q is cheaper than the replicating portfolio, then you short the replicating portfolio use some of the proceeds to buy Q and keep the rest. Tomorrow (time T), you get the Q payout, which exactly matches the portfolio you have to buy to undo your short portfolio position.

We find the portfolio weights w_i by solving a system of linear equations (1) in terms of the payouts Q_j. Because the equations are linear, the w_i are linear functions of the Q_j. Therefore the price today, which is the sum of the w_i is also a linear function of the Q_j:

$$\text{Price}(Q) = \sum_{i=1}^{n} w_i = \sum_{j=1}^{n} s_j Q_j \; . \tag{2}$$

Even before we know how to find the numbers s_j, we know they exist. That's because a linear function of the Q_j can be written as a sum like this. The market, as determined by the price matrix P_{ij} is *arbitrage free* if $s_j \geq 0$ for all j. If $s_j < 0$ for any j, then there is an arbitrage, as we will soon see.

But first the linear algebra in more detail. The equations (1) may be written in terms of the vectors Q and w:

$$Q = \begin{pmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{pmatrix} \quad , \quad w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \; .$$

The equations take the form

$$P^t w = Q \; .$$

It would be Pw rather than $P^t w$ if it were $P_{ij} w_j$ rather than $P_{ij} w_i$. The solution is

$$w = \left(P^t\right)^{-1} Q \; .$$

It is a fact of linear algebra that the inverse of the transpose is the same as the transpose of the inverse:

$$\left(P^t\right)^{-1} = \left(P^{-1}\right)^t \ .$$

The portfolio price today is expressed in terms of the **1** vector whose entries are all ones:

$$\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \ .$$

The formula is

$$\sum_{i=1}^{n} w_i = \mathbf{1}^t w \ .$$

Therefore, the price of the replicating portfolio today is

$$\begin{aligned} \sum_{i=1}^{n} w_i &= \mathbf{1}^t w \\ &= \mathbf{1}^t \left(P^t\right)^{-1} Q \\ &= \left[\mathbf{1}^t \left(P^t\right)^{-1}\right] Q \\ &= \left[\mathbf{1}^t \left(P^{-1}\right)^t\right] Q \\ &= \left[P^{-1}\mathbf{1}\right]^t Q \end{aligned}$$

The last line here is in the form of the last part of (2). We make this more explicit by defining the vector $s = P^{-1}\mathbf{1}$. Then $s^t Q = \sum_j s_j Q_j$, which is exactly (2). This shows that we find the *multipliers* s_j in (2) by

$$s = P^{-1}\mathbf{1} \ .$$

The s_j are almost, but not quite, the risk neutral prices.

We want to interpret the multipliers s_j in terms of risk neutral probability and the risk free rate. For that purpose, look at the portfolio that pays $Q_j = 1$ for all j. There already is an asset like this in the market, it is B times asset 1. This is because asset 1 pays $\frac{1}{B}$ for all j. Therefore, the price of this "option" today is B times the price of asset 1, which is B. Since each of the Q_j is equal to 1, and the price today is B, the general pricing formula (2) becomes

$$B = \sum_{j=1}^{n} s_j \ .$$

In particular, we learn that at least one of the s_j is positive, whether or not there is an arbitrage opportunity. The risk neutral probabilities are

$$r_j = \frac{1}{B} s_j \ .$$

Assume, for now, that $s_j \geq 0$ for all j. Then $r_j \geq 0$ for all j and $\sum r_j = 1$. For this reason, we may *interpret* the r_j as probabilities. Any set of non-negative numbers that add up to 1 may be interpreted as probabilities. In finance, these are called *risk neutral* probabilities. The price today of the option that pays Q_j is

$$\text{Price}(Q) = \sum_{j=1}^{n} s_j Q_j = B \sum_{j=1}^{n} r_j Q_j = B\text{E}_{\text{RN}}[Q_j] \ .$$

The term *risk neutral* refers to the "risk neutral" investor who doesn't care whether an asset is risky or not. She values a risk asset at its expected value. Risk neutral pricing says that the price today of the option Q is the same as it would be if every investor were risk neutral (most are risk *averse*, they don't like risk) and the probability of state j were r_j (it isn't). You take the expected value tomorrow, which is $\text{E}_{\text{RN}}[Q_j]$, and multiply by the discount factor B, which is the price today of one unit of currency tomorrow. That's risk neutral pricing.

The signs of the s_j, which are the same as the signs of the r_j determine whether there is an arbitrage in the market. If there is a negative s_j, then there is an arbitrage. If $s_j \geq 0$ for all j, then there is no arbitrage and we have true risk neutral pricing. Suppose s_k is negative for some k. We saw that at least one of them is positive, so suppose $s_l > 0$. Consider the option that pays 1 in state k and $\frac{-s_k}{s_l}$ in state l, and zero in all other states. Because P is non-singular, there is a portfolio of market assets that replicates this payout. This payout is never negative, and it is positive in states k and l (note: $\frac{-s_k}{s_l} > 0$ because $s_k < 0$ and $s_l > 0$). The price today of this portfolio is

$$\sum_{i=1}^{n} w_i = \sum_{j=1}^{n} s_j Q_j = (\, s_k \cdot 1\,) + \left(\, s_k \cdot \frac{-s_k}{s_l} \,\right) = 0 \ .$$

This is the definition of arbitrage: it's a portfolio of market assets that has cost zero today and positive payout tomorrow (more technically, the payout is never negative and sometimes positive). Therefore, if there is a negative s, there is an arbitrage in the market. Now, suppose $s_k = 0$, then look at the portfolio that pays 1 in state k and zero in every other state. Its price today is $s_k \cdot 1 = 0$ and its payout is sometimes positive. That's also an arbitrage.

On the other hand, suppose all the s_k are strictly positive. Let the payout amounts, Q_j be non-negative and positive for at least one j, Then the price today of that portfolio is

$$\sum_{j=1}^{n} s_j Q_j > 0 \; .$$

It isn't an arbitrage.

Those who are double-majoring in economics may have seen a version of this argument. In economics it is common to consider the *Arrow Debreu security*, which pays one unit in state j and zero in any other state. The numbers s_j are the *Arrow Debreu prices* Equation (2) shows that s_j is what it costs to buy the "option" that pays one in state j and zero in all other states.

In specific examples you can take shortcuts or do the calculations informally. Question 6 on the midterm had the following table and asked you to calculate the risk neutral probabilities or show that there is an arbitrage opportunity.

	asset	A	S	C
state				
1		2	1	0
2		2	2	0
3		2	3	10

Let Π_t be the value of the portfolio at time t. Suppose Π is $w_1 A + w_2 S + w_3 C$. Consider an option that pays Q_1, Q_2, or Q_3 in states 1, 2, and 3 respectively. The price today of Q is Π_0, where Π is the replicating portfolio. There is a replication equation for each of the three possible states at time T. These are

$$\begin{array}{lll} \text{state 1:} & Q_1 & = w_1 \cdot 2 + w_2 \cdot 1 \\ \text{state 2:} & Q_2 & = w_1 \cdot 2 + w_2 \cdot 2 \\ \text{state 3:} & Q_3 & = w_1 \cdot 2 + w_2 \cdot 3 + w_3 \cdot 10 \end{array}$$

We can use Gaussian elimination (you can call it just "elimination") to solve for the weights in terms w_i in terms of the payouts Q_j. Subtract equation 1 from equation 2 and you get

$$Q_2 - Q_1 = w_2 \; .$$

Subtract $2 \times$ (eqn 1) from (eqn 2) and you get

$$Q_2 - 2Q_1 = -2w_1 \implies w_1 = Q_1 - \frac{1}{2}Q_2 \; .$$

Insert the known formulas for w_1 and w_2 into (eqn 3) and you get

$$Q_3 = 2\left(Q_1 - \frac{1}{2}Q_2\right) + 3\left(Q_2 - Q_1\right) + 10w_3 \; .$$

This gives

$$w_3 = \frac{1}{10}\left(Q_1 - 2Q_2 + Q_3\right) \ .$$

The price of the Q option today is

$$\begin{aligned}
\Pi_0 &= w_1 + w_2 + w_3 \\
&= \left(Q_1 - \frac{1}{2}Q_2\right) + \left(Q_2 - Q_1\right) + \left(\frac{1}{10}Q_1 - \frac{1}{5}Q_2 + \frac{1}{10}Q_3\right) \\
&= \frac{1}{10}Q_1 + \left(\frac{1}{2} - \frac{1}{5}\right)Q_2 + \frac{1}{10}Q_3 \ .
\end{aligned}$$

This has the form $s_1Q_1 + s_2Q_2 + s_3Q_3$ with $s_1 = \frac{1}{10}$, $s_2 = \frac{1}{2} - \frac{1}{5}$, and $s_3 = \frac{1}{2}$. The risk free asset is asset A. Its value at time T is 2, so the "bond price" is $B = \frac{1}{2}$. The theory says that $s_1 + s_2 + s_3 = B$, which is true in this case: $\frac{1}{10} + \left(\frac{1}{2} - \frac{1}{5}\right) + \frac{1}{10} = \frac{1}{2}$. All the s_j are positive so there is no arbitrage opportunity. The risk neutral probabilities are

$$\begin{aligned}
r_1 &= \frac{1}{B}s_1 = 2 \cdot \frac{1}{10} = \frac{1}{5} \\
r_2 &= \frac{1}{B}s_2 = 2 \cdot \left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{5} \\
r_3 &= \frac{1}{B}s_3 = 2 \cdot \frac{1}{10} = \frac{1}{5}
\end{aligned}$$