Midterm exam, October 28

(small errors corrected)

Instructions:

- Explain your reasoning. Points may be subtracted even for correct answers otherwise.
- Cross out anything you think is wrong. Points may be subtracted for wrong answers even if the correct answer also appears.
- You will receive 20% credit for a blank answer. You may have points deducted from this for a wrong answer.
- Answer each question in the space provided.
- You are allowed one "cheat sheet", which is a standard size piece of paper with whatever you want written on it.

1. Explain how to replicate a put option with strike price K using a portfolio that contains call options on the same underlier with the same strike price, the underlier itself, and a risk free asset. Draw a diagram to illustrate your construction. The diagram should contain the put payout, the call payout, and the underlier.

2. The quadratic form $f(x_1, x_2) = x_1^2 + (x_2 - x_1)^2$ may be written in the form $f(x_1, x_2) = x^t Ax$, where $x = (x_1, x_2)^t$, and A is a 2×2 symmetric matrix.

   (a) Find A.

   (b) Determine whether A is positive definite.

   (c) The n variable version of this function is

   $$f(x_1, \ldots, x_n) = x_1^2 + (x_2 - x_1)^2 + \cdots + (x_n - x_{n-1})^2 \; .$$

   This can be represented using n n component vector $x = (x_1, \ldots, x_n)^t$, and a symmetric $n \times n$ matrix as $f(x_1, \ldots, x_n) = x^t Ax$. Is this A also positive definite? Explain your reasoning without necessarily giving the form of A.

3. Supose $X \sim \mathcal{N}(\mu, \sigma^2)$ and $S = e^{rX}$.

   (a) Write an integral that represents the $E[S]$ as a function of the parameters r, μ, and σ^2. This expression should be an integral over x involving the PDF of X. Do not evaluate the integral for this part.

   (b) Evaluate the integral to get an explicit formula for $E[S]$. Hint: The integral involves an exponential that involves x^2 and x. Complete the square to put this in the form

$$e^{\frac{-(x-a)^2}{2\sigma^2}+b},$$

   where a and b depend on the parameters but not on x.

   (c) Show that the formula from part (b) has the limit as $\sigma^2 \to 0$.

4. Consider a simple mortgage loan with the following features

- The holder of the mortgage (the one who lent the money) receives continuous coupon payments with rate c up to a *prepayment* time T.
- At time T, the holder receives the principal, P.
- The prepayment time is an exponential random variable with intensity parameter λ. A payment at time t is discounted by the factor e^{-rt}.

(a) Determine the continuous interest rate, μ, under which a payment of $c\,dt$ for a time interval dt on principal P leaves the principal unchanged. Normally, the outstanding principal at time $t + dt$ would be the outstanding principal at time t plus the interest for time dt minus the amount paid at time t for interval dt.

(b) Determine the expected value of the total discounted value of all the payments.

(c) Under what circumstances (what combinations of μ and r) would the holder prefer that the prepayment intensity is lower?

5. Suppose X_1, X_2, and X_3 are three assets with means $E[X_j] = \mu_j$ and covariances $\text{cov}(X_j, X_k) = C_{jk}$. Suppose w is a set of proposed portfolio weights. Specifically, take

$$\mu = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad , \qquad C = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 6 \end{pmatrix} \quad , \qquad w = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} .$$

   (a) Use vector and matrix notation to give expressions for the portfolio expected value, μ, and the portfolio variance, σ^2.

   (b) Show that the proposed portfolio weights are not efficient.

   (c) Consider perturbations of the portfolio weights Δw_1, Δw_2, Δw_3. Let $\Delta\mu$ and $\Delta\sigma^2$ be the corresponding perturbations in the portfolio expected value and variance. Write approximate expressions for $\Delta\mu$ and $\Delta\sigma^2$ that are accurate to first order in Δw. Hint: It may be easier to work with the vector of perturbations

$$\Delta w = \begin{pmatrix} \Delta w_1 \\ \Delta w_2 \\ \Delta w_3 \end{pmatrix} .$$

   (d) Find specific small values of Δw_1, Δw_2, Δw_3 that satisfy the budget constraint exactly and do not change μ, but, to first order approximation,reduce σ^2.

6. Suppose there are three assets, A (a bond), S (a stock), and C (a call). The price of each asset today is 1. There are three possible states "tomorrow", and the corresponding prices are in the table.

	asset	A	S	C
state				
1		2	1	0
2		2	2	0
3		2	3	10

(a) Show that this a complete and arbitrage free market.

(b) Calculate the risk neutral probabilities, q_1, q_2, and q_3 of the three states.

(c) Find the price today of the option P that pays 1 "tomorrow" if $S = 1$, and 0 if $S > 1$.

7. Suppose X and Y are two correlated random variables that take values $\{1, \ldots, n\}$. Suppose $p_{ij} = \Pr(X = i \,.\, Y = j)$. Assume the numbers p_{ij} are already stored in a matrix `p`, whose values are `p[i,j]`. Write a program in R that computes $\operatorname{cov}(X, Y)$. Use one or more double loops.