Sample questions for October 28, whole period midterm exam. Also review the quiz and sample quiz questions

Instructions (for the midterm):

- Explain your reasoning. Points may be subtracted even for correct answers otherwise.
- Cross out anything you think is wrong. Points may be subtracted for wrong answers even if the correct answer also appears.
- You will receive 20% credit for a blank answer. You may have points deducted from this for a wrong answer.
- Answer each question in the space provided.
- You are allowed one "cheat sheet", which is a standard size piece of paper with whatever you want written on it.

Questions (the midterm will be shorter than this):

1. Suppose portfolio P consists of two calls stuck at $K = 10$ and one put struck at $K = 10$. Draw a graph of the value $P(X)$ for X in the range $0 \leq X \leq 20$.

2. The simple puts and calls we have been discussing are often called *vanilla* options, or even *plain vanillas*. A *capped call* is a call option with a cap on the possible value. A capped call pays the call value as long as that is less than the maximum. Otherwise it pays the maximum. Suppose $V(x)$ is a capped call struck at $K = 5$ and a maximum payout of 10.

   (a) Draw a graph of the payout of this capped call.

   (b) Explain how to replicate the payout with a combination of a vanilla call and a vanilla put.

3. The normal probability density $\mathcal{N}(\mu, \sigma^2)$ is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} .$$

   Suppose μ is replaced by $\mu + \Delta\mu$ and σ is replaced by $\sigma + \Delta\sigma$. That is, suppose we make small changes (perturbations) to μ and σ. Find the relation between $\Delta\mu$ and $\Delta\sigma$ so that $\Delta f = 0$, to leading order. This is not an exact relation, but a linear approximate relation.

4. Suppose X_1 and X_2 are financial assets with expected values $\mathrm{E}[X_1] = 1$ and $\mathrm{E}[X_2] = 2$. Suppose $\mathrm{var}(X_1) = 1$, $\mathrm{var}(X_2) = 2$, and $\mathrm{cov}(X_1, X_2) = 1$.

   (a) Identify the minimum variance portfolio and its expected return.

   (b) Make a graph of the efficient frontier.

5. Suppose there are financial assets whose values are $X_1, \ldots, X_n$ and the minimum variance portfolio has variance σ_1^2. Now suppose the portfolio is allowed also to include a put option on asset X_1 and a call option on asset X_2.

   (a) Let σ_2^2 be the minimum variance of a portfolio containing these $n+2$ assets. Is $\sigma_2^2 > \sigma_1^2$, $\sigma_2^2 < \sigma_1^2$, or $\sigma_2^2 = \sigma_1^2$? Which of these three is impossible. Which is barely possible but unlikely in the real world?

   (b) Use the answer to point (a) to give a motivation for including options in an investment portflio.

6. An *annuity* is a financial instrument that pays a fixed coupon until it expires. Unlike a coupon paying bond, an annuity does not make a principal payment at the expiration time. A risky annuity has default intensity λ, continuously payed coupon rate c, and expiration time D. The default time is modeled as an exponential random variable.

   (a) In terms of these parameters, find a formula for the expected total payout.

   (b) Suppose payments at time t are discounted by a factor e^{-rt}. What then is the expected sum (integral) of the present values of all the payments for this risky annuity?

7. The value of asset X is unknown, but modeled as a Gaussian with mean μ and variance σ^2.

   (a) Use the cumulative normal distribution function $N(x)$ to give a formula for $\Pr(X > K)$.

   (b) Suppose $V(X)$ is the value of a call option struck at K. Use $N(x)$ to give a formula for $\mathrm{E}[V(X)]$. Hint: Write the integral for the expectation, which involves x and the Gaussian probability density. Use the substitution $y = x - \mu$ in the integral, which turns it into two integrals. Do one of them be explicit integration and get an answer involving the exponential function. Express the other one in terms of the N function.

8. Use the Cholesky factorization algorithm to determine whether the matrix is positive definite. If it is, find the Cholesky factor.

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 5 & 3 \\ 1 & 3 & 14 \end{pmatrix}$$

9. Suppose X and Y are two correlated random variables that take values $\{1, \ldots, n\}$. Suppose $p_{ij} = \Pr(X = i\,.\,Y = j)$. The *marginal* probabilities for X are

$$q_i = \Pr(X = i) = \sum_{j=1}^{n} \Pr(X = i\,,\ Y = j)\ .$$

Write a program in R that computes these marginal probabilities from the joint probabilities p_{ij}. Assume the numbers p_{ij} are already stored in a matrix `p`, whose values are `p[i,j]`. Do not assume the array/vector `q` has been allocated. Use a double loop. Write a program in `R` to compute the mean and variance of X. Use either a single or a double loop.