Mathematics of Finance, Courant Institute, Fall 2015

http://www.math.nyu.edu/faculty/goodman/teaching/mathFin/index.html

Always check the class message board before doing any work on the assignment.

Assignment 2, due September 23

Corrections: ((3a) changed a little. Parentheses in the formula of (4) fixed. See message board)

1. Suppose an investment that returns 10%/year. This investment has value $V(t) = e^{rt}$ so that $V(1) = 1.1 \cdot V(0)$. (The continuous interest r is in percent/year and t is in years.) This assumes that the return is compounded continuously and reported annually. Think about whether $r > .1$ or $r < .1$. If you had a return of 10%/year that was not compounded, your investment would double in ten years. How long does it take for this continuously compounded investment to double? Give the exact answer in a formula, such as: "Doubling time $= T_d = \sqrt{10}$." (not the correct formula). Then give a reasonably accurate number, such as: "Doubling time $= T_d = 3.162$ years." Then give a round number that contains the important takeaway information, such as "The actual doubling time, with compounding, is a little more than three years." As an applied mathematician, you will have to present information in the form most appropriate to the person you are talking to.

2. Consider a "bond" with the following structure. The investor pays one unit of money at time $t = 0$. After one period, the investor receives a "coupon" payment c. After a second period of the same length, the investor receives the principal and the bond is ended. Suppose the one period return, not continuously compounded, is r. This means that one unit today gives $1 + r$ after one period, and $(1 + r)^2$ after two periods. This exercise is about finding the value of r implicit in the above numbers. This is the *yield to maturity* of the bond.

   (a) Write a formula that expresses the fact that the present value at time $t = 0$ (today) of the bond payments is equal to the price today.

   (b) Use this to find a formula for r in terms of c. Hints: It involves the quadratic formula. It may be easier to write a formula for $V = 1 + r$ first.

   (c) Find a simple approximation to this formula that is valid when c is small. Find the formula applying Taylor series approximation to the exact formula. It will take the form $r \approx Mc$, where M is a *multiplier*. As an applied mathematician, you will have to present simple approximate formulas that hold in real world situations rather than complicated but exact formulas that are also true in situations that don't come up in practice.

3. A *coupon bond* pays a *coupon* amount, c, at regular times, and the principal, P, at the end. If there are n coupon payments separated by Δt, then coupon k is paid at time $t_k = k\Delta t$. There is no coupon payment at $t = 0$ and no payment at the maturity time $T = (n+1)\Delta t$. The payments are at times $t_1, t_2, \ldots, t_n$. If the continuous compounding interest rate is r, then the *present value* of all these payments is

$$\text{PV} = \sum_{k=1}^{n} e^{-rt_k} c + e^{-rT} P \; . \tag{1}$$

   (a) The sum in (1) is a geometric series, so there is a formula for it. To get this formula, define $z = e^{-r\Delta t}$. Show that $z^k = e^{-rt_k}$. The sum is $S = z + z^2 + \cdots z^n$. One form of the famous trick for doing the sum is to write $\frac{1}{z}S = 1 + z + \cdots + z^{n-1}$. You $\frac{1}{z}S - S = 1 - z^n$. Use this to get a formula of the form $\text{PV} = \cdots$ involving r, c, P, and T (or Δt and T or ...). It isn't a complicated formula. You could use n to express the formula, but it is better not to. That is because it might not be clear how to do part (b).

   (b) It might be easier to quote a *coupon payment rate* rather than the coupons themselves. The rate, R, is the amount of coupon payment per year. If Δt is in years (say $\Delta t = \frac{1}{4}$ for quarterly payments), then the the individual payments are related to the rate by

$$c = \Delta t R \; . \tag{2}$$

   Express your answer to part (a) in terms of R instead of c. Find the limit of PV in the limit $\Delta t \to 0$ with T, r, and R fixed. This is the continuous coupon payment approximation.

   (c) What is the error in the continuous coupon payment approximation if $R = 5\%/\text{year}$ (this means 5% of P, which you can take to be $\$10,000$), payments are twice per year, T is 30 years, and $r = 4\%/\text{year}$? Express the error in dollars (present value) and in a percentage of the true present value.

   (d) There is a simpler way to get the continuous coupon present value. Substitute (2) into (1), and notice that the sum looks like a Riemann sum approximation to an integral. Write the integral, work the integral, and you should get the formula you got before. The difference is that you don't have to know about geometric series this way.

4. Suppose interest rates are low today but expected to increase in the future. A simple model of that is $r(t) = r_0 + st$. Here r_0 is the spot rate today and s is the *slope* (a technical term in finance). Consider a *money market* account that gets interest payments at times $t_k = \Delta t$. At the end of the period, the account gets the interest rate at the beginning of the period. Let V_k be the value of the account at time t_k, just after the interest for

the previous period is paid. The starting value is V_0. The relations are

$$
\begin{aligned}
V_1 &= (1 + \Delta t\, r_0) V_0 \\
V_2 &= (1 + \Delta t\, r(t_1)) V_1 = (1 + \Delta t\, r(t_1))(1 + \Delta t\, r_0) V_0 \\
\vdots\; &\;\vdots \qquad\qquad \vdots \\
V_n &= \left(\prod_{k=0}^{n-1} (1 + \Delta t\, r(t_k)) \right) V_0 = B_n V_0
\end{aligned}
$$

Find the limit of B_n as $\Delta t \to 0$ and $n \to \infty$ with $T = n\Delta t$ fixed (so $\Delta t = T/n$). Hint: Calculate $\log(B_n)$, which is a sum of logs. Use Taylor approximations to approximate the logs in the sum as accurately as necessary. Recognize the resulting sum as the approximation of an integral. Calculate the integral.

5. A mortgage has payments P that are made at times $t_k = k\Delta t$. It has an annual interest rate r that is added to the balance at times t_k. Suppose the balance starts at V_0, until the balance is zero, it is updated using

$$
V_{k+1} = (1 + r\Delta t) V_k - P \;. \tag{3}
$$

The balance gets larger because of interest and it gets smaller because of the payment. The mortgage will never be paid off unless $P > r\Delta t V_0$, which we assume. Write a program in `R` that asks the user for the parameters (interest rate, number of payments per year, the size of each payment (P), and the initial balance), and returns the time, in years, until the balance is zero. This should involve a `for` loop with an `if` test and a `break`. You can learn about these things from `Loops.pdf` and `Loops.R`, which are on with this assignment. Try to copy (*re-use* is the technical term) as much code as possible from `Loops.R` and from last assignment. Your code must check, before the loop, whether it will ever stop. If P is to small to ever pay the balance, give the user an error message and do not execute the loop.